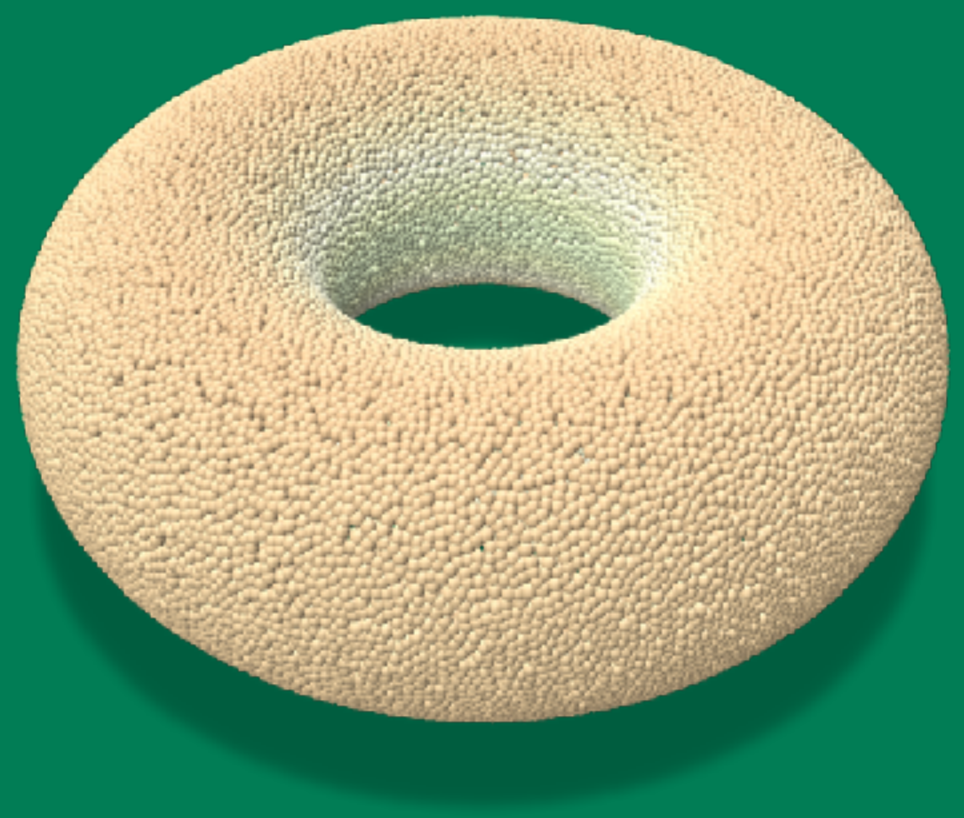
Exploring the Laplacian in Computer Graphics

Week 4



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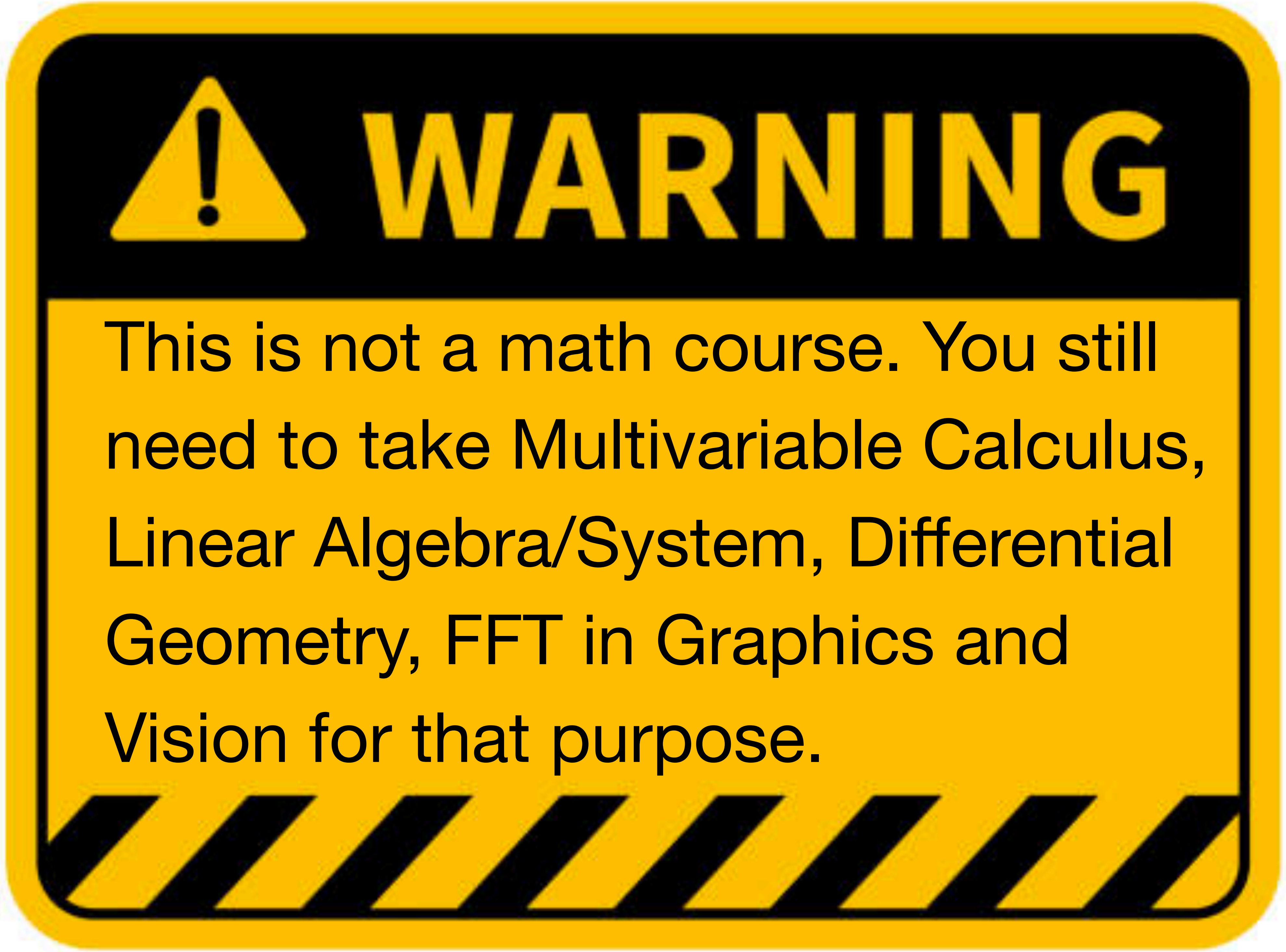
The Johns Hopkins University

2023 Fall

Math Preliminaries



This is not a math course. You still need to take Multivariable Calculus, Linear Algebra/System, Differential Geometry, FFT in Graphics and Vision for that purpose.

Math Preliminaries

"Matrix"

A collection of numbers, organized neatly in a grid/table

e.g.

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

This is what we call a 3 by 3 matrix

Math Preliminaries

Where can a function f live?

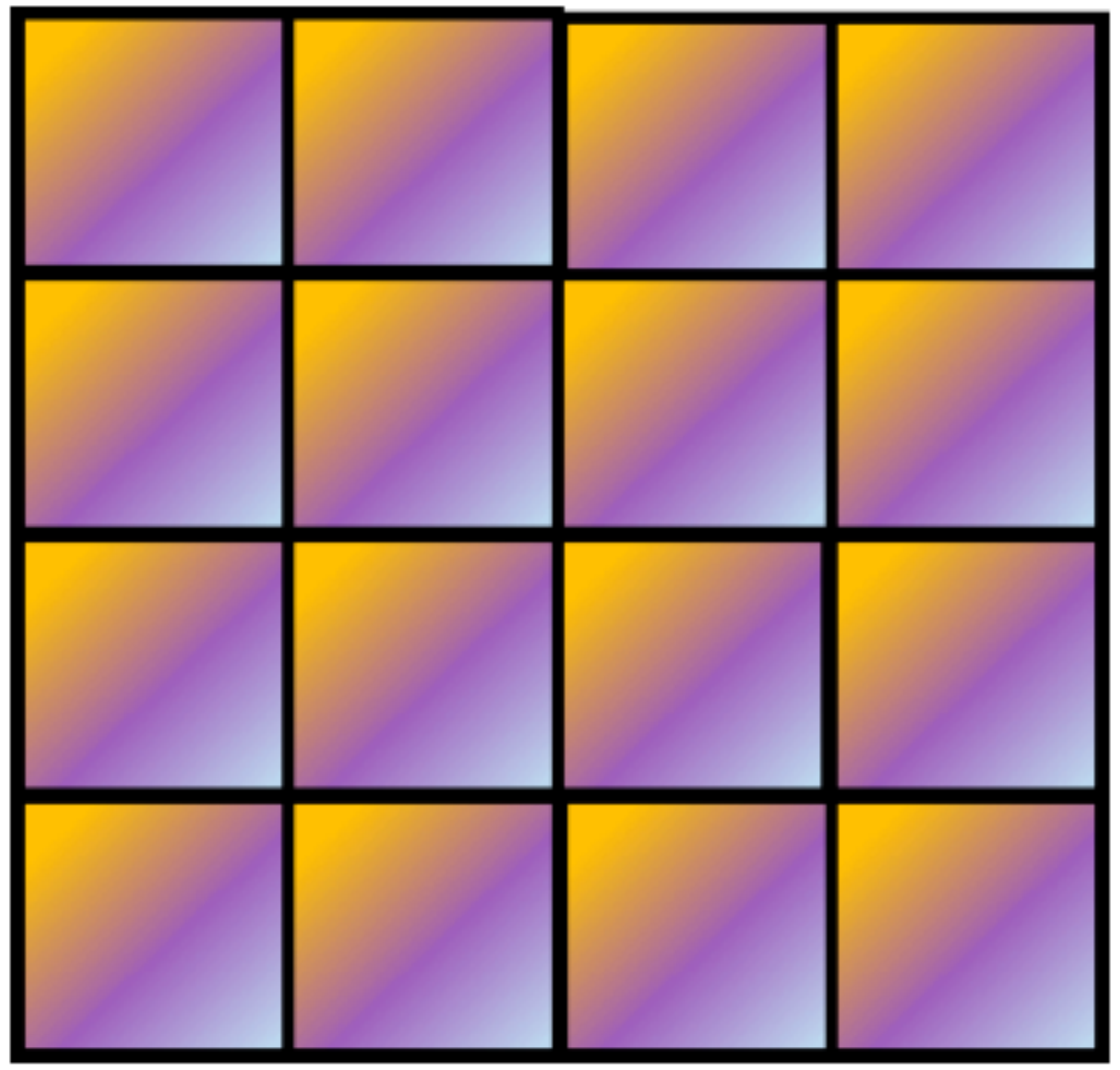
From high school: $f(x) = x^2$

How about these? $f(x, y) = x^2 + y^2$

$$f(x, y, z) = x^2 + y^2 + z^2$$

Where can a function f live on the 2D grid?

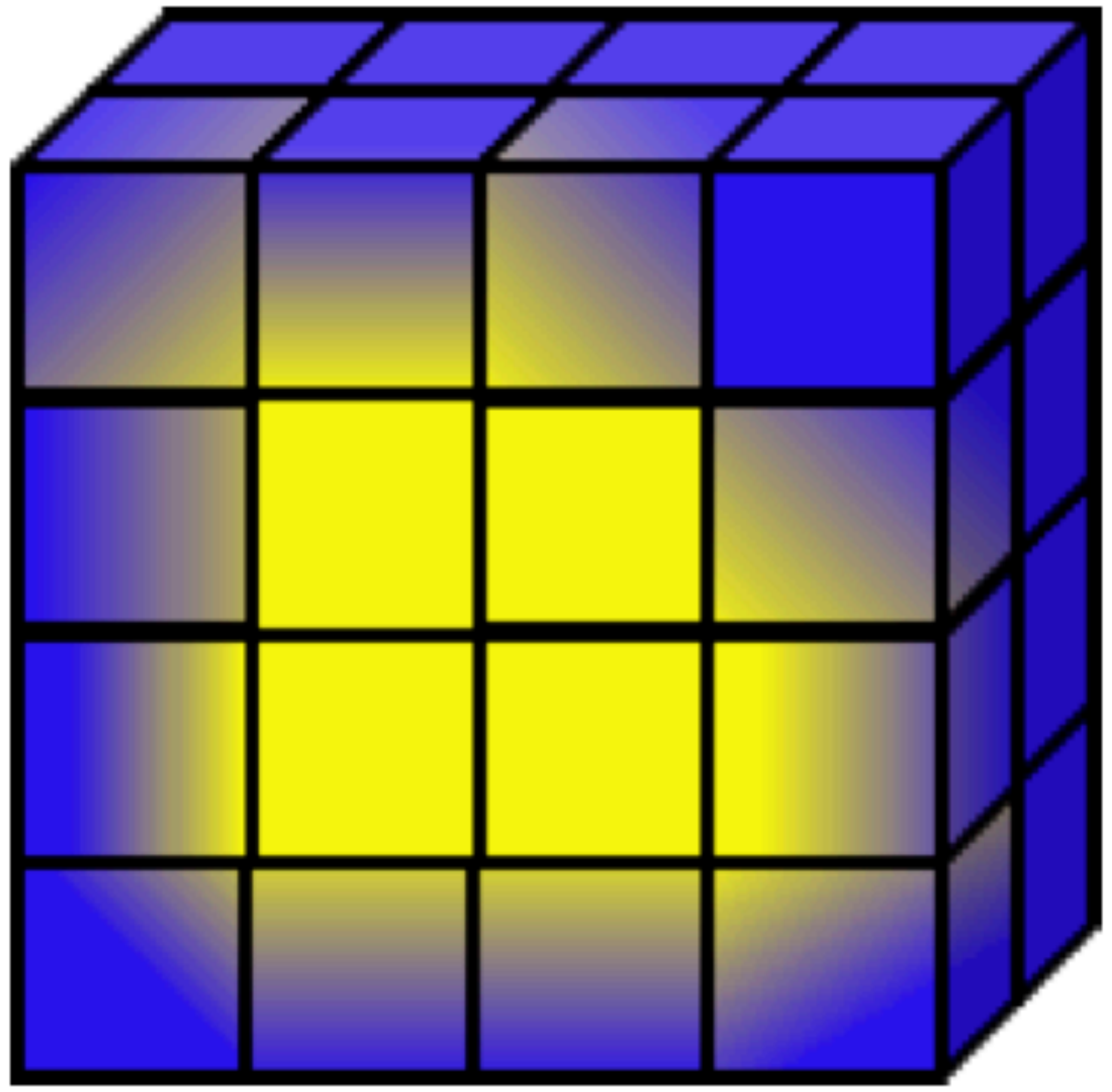
$$f(x, y) = x^2 + y^2$$



Math Preliminaries

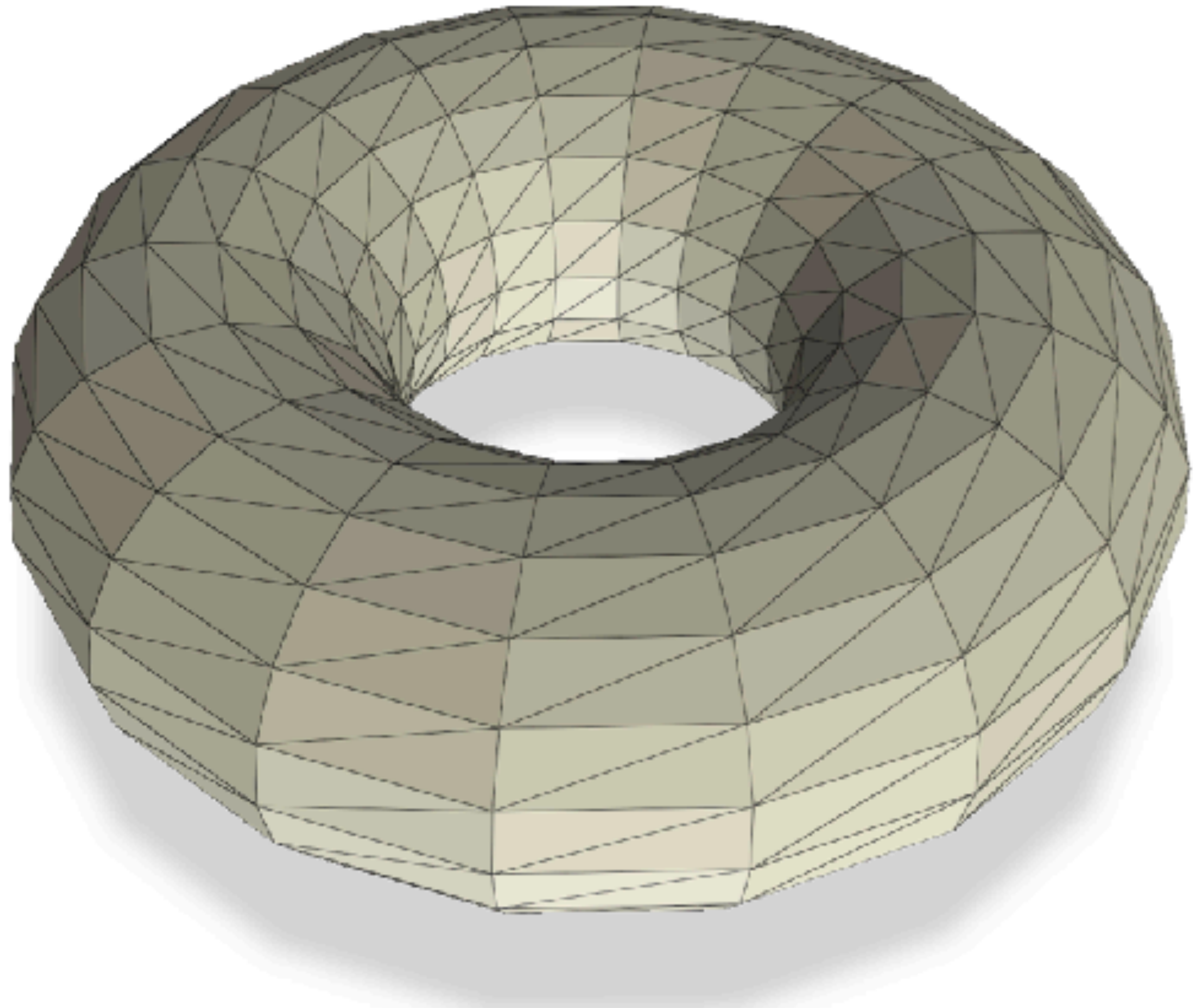
Where can a function f live on the 3D grid?

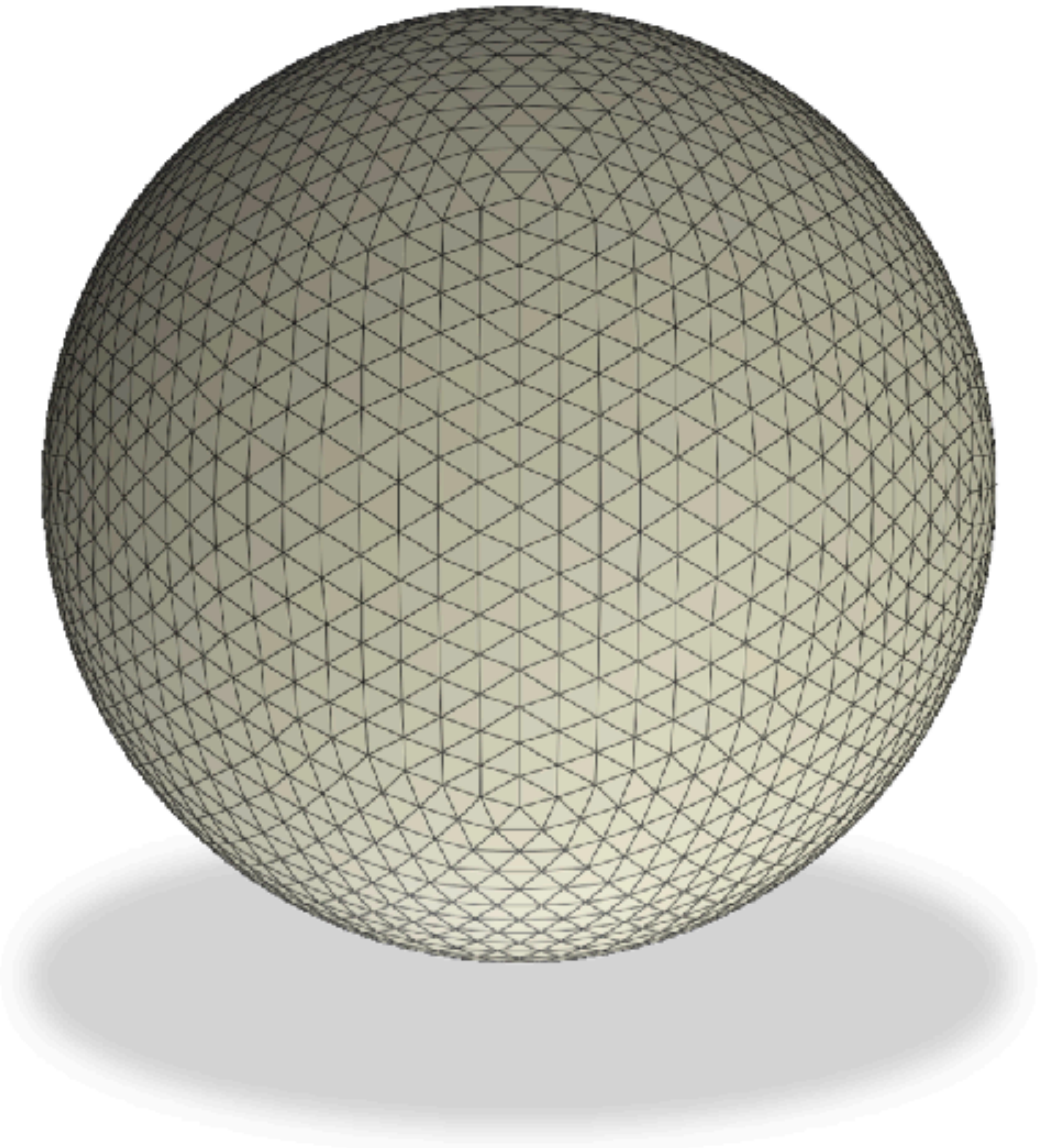
$$f(x, y, z) = x^2 + y^2 + z^2$$



Where can a function f live on the triangle mesh?

$$f(x, y, z) = x^2 + y^2 + z^2$$





What is the Laplacian

Δ The Laplacian

A tool for signal processing!

1D: audio processing, geometry processing

2D: image processing, geometry processing

3D: geometry processing

What is the Laplacian

Δ The Laplacian

A function $f : \mathbb{R}^n \to \mathbb{R}$ $\longrightarrow$ $\boxed{\Delta}$ $\longrightarrow$ A function $\Delta f : \mathbb{R}^n \to \mathbb{R}$

an operator

What is the Laplacian

one dimensional

continuous setting

$$\Delta f(t) = \frac{d^2 f(t)}{dt^2}$$

discrete setting

$$\Delta f[t] = f[t+1] - 2f[t] + f[t-1]$$

* example stolen from Justin Solomon

What is the Laplacian

one dimensional (audio processing)

You have an audio signal

$f(t)$ = **amplitude of the sound at time t**

$$\Delta f[t] = f[t+1] - 2f[t] + f[t-1]$$

Q: When would $|\Delta f[t]|$ be large?

A: rapid change of the sound, e.g. a drum hit

What is the Laplacian

one dimensional (geometry processing)

You have a pearl necklace
with pearls of graduated sizes

$f(t) =$ **size of pearl t**

$$\Delta f[t] = f[t+1] - 2f[t] + f[t-1]$$

Q: When would $|\Delta f[t]|$ be large?

A: when the size of pearl t is significantly from its neighbors!

What is the Laplacian

two dimensional (image processing)

Q: How are images stored in computer? (we learned this in Week2)

A: regular grid

Q: What's the dimension? (we learned this in Week2)

A: height by width by 3

Q: If the Laplacian can highlight rapid change, how could it help enhancing your image?

A: image sharpening!

What is the Laplacian

two dimensional (image processing)

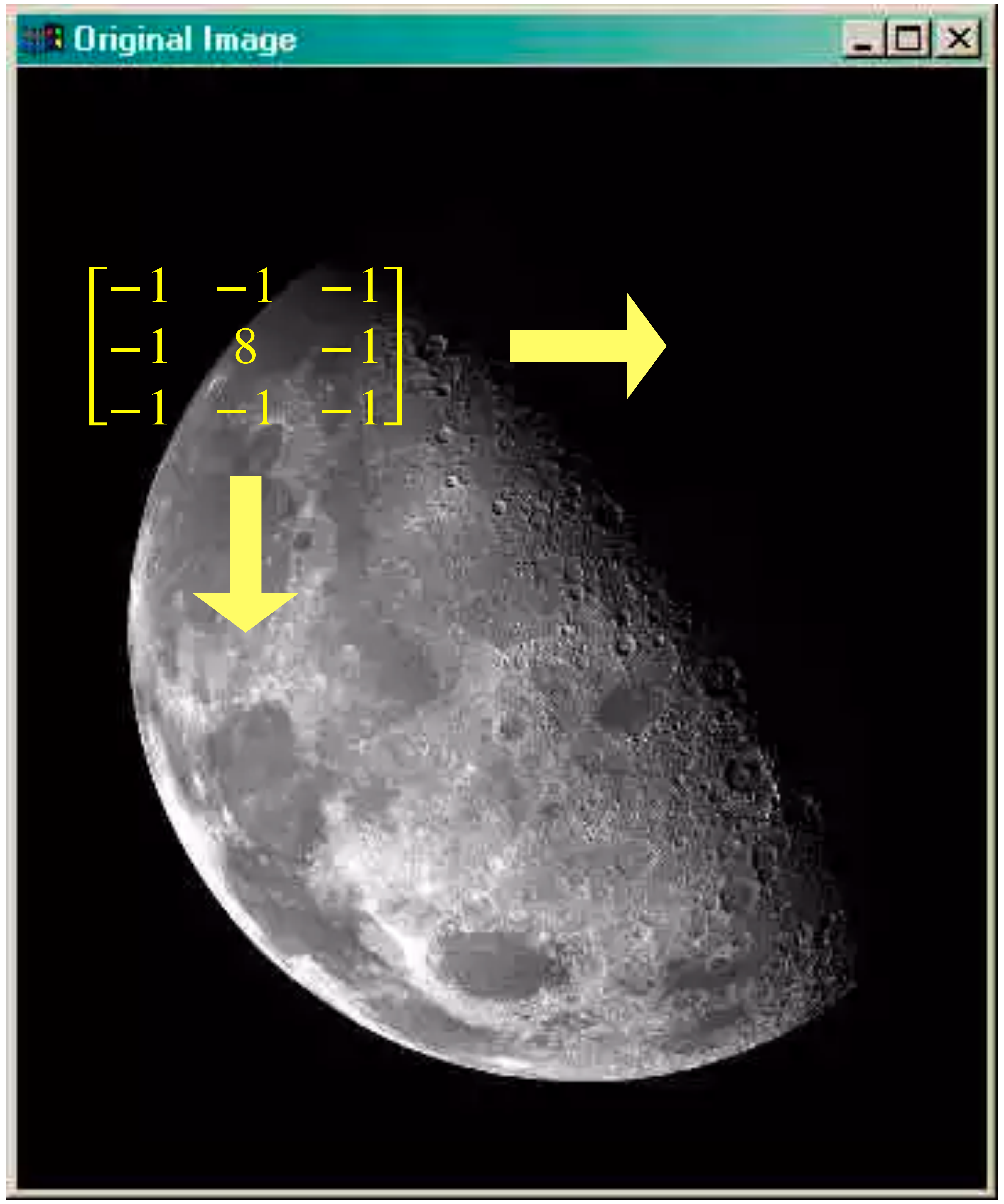
$f(i,j) =$ **pixel value**

$$\Delta f(i,j) = \frac{\partial^2 f}{\partial i^2} + \frac{\partial^2 f}{\partial j^2}$$

The "stencil" / "kernel" $\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix}$

What is the Laplacian

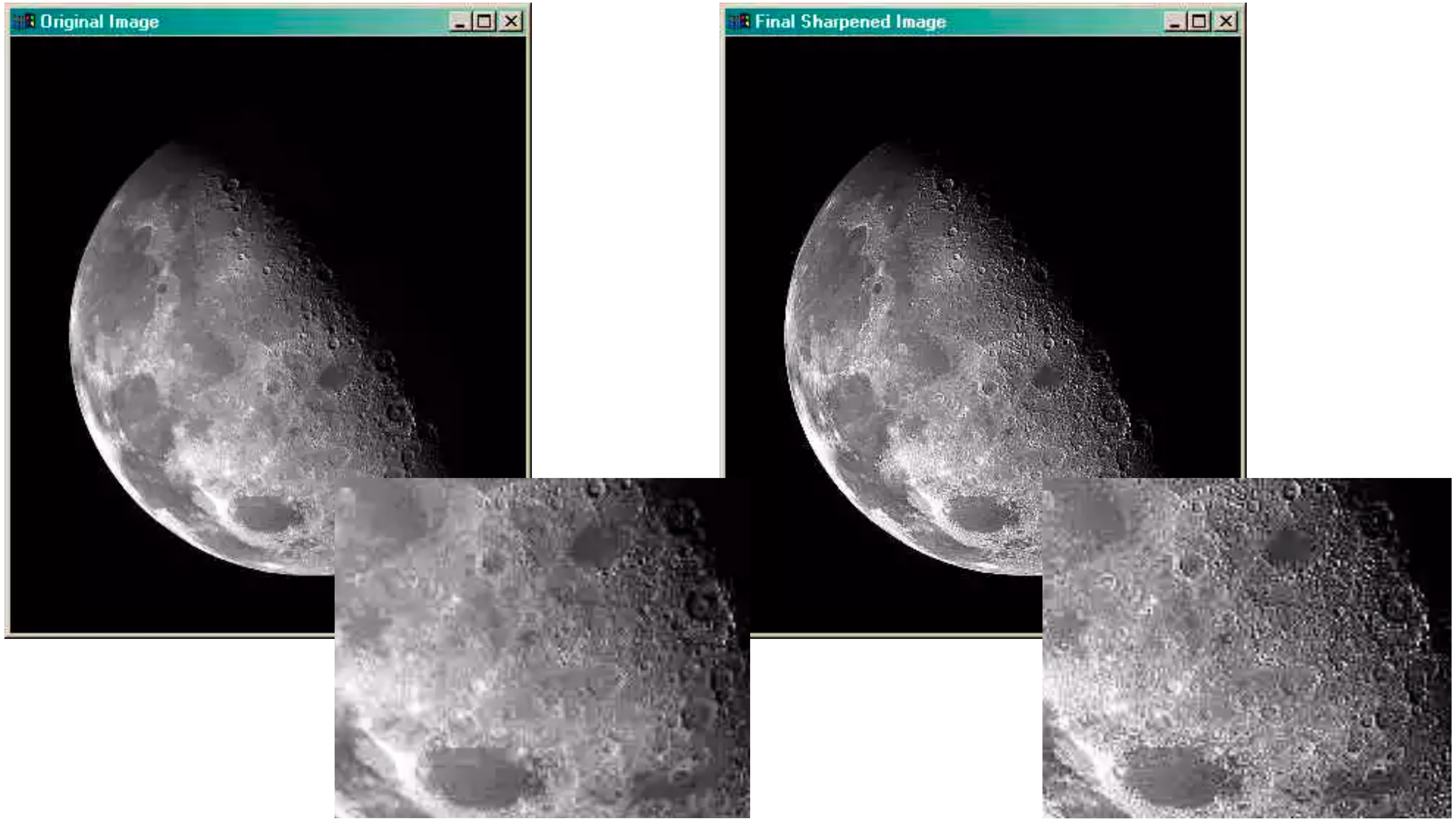
two dimensional (image processing)



1. Sweep the “stencil” / “kernel” across the image
2. As you sweep, conduct a calculation called “convolution” between the stencil and the patch of that image.
3. You get sth highlighting the rapid change, we call that “edge map”
4. Add the edge map back to the original image, and get the sharpened image

What is the Laplacian

two dimensional (image processing)



http://www.idlcoyote.com/ip_tips/sharpen.html

What is the Laplacian

two dimensional (image processing)

Note that, if you do the sharpening on gradient domain, it gives you better results.
But gradient-domain-processing a more advanced topic, covered in 601.457.

before

after

http://grail.cs.washington.edu/projects/gradientshop/demos/gs_paper_TOG_2009.pdf

What is the Laplacian

three dimensional (geometry processing)

Where does function f live?

imagine you are knitting a turtle……and you are sewing beads (function values) at the knitting cross (vertices)

* image credit: Adobe Stock

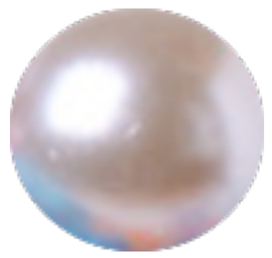
What is the Laplacian

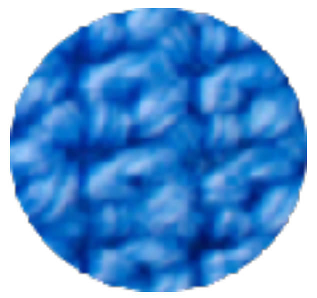
three dimensional (geometry processing)

scalar function living on vertices: beads

Laplacian: helps us understand how each bead is different from its neighboring beads

“quality inspector”
of beads distribution





beads	**“bridge”**	knit
(the scalar function)		(the shape)

* image credit: Adobe Stock

What is the Laplacian

three dimensional (geometry processing)

With Laplacian, now you have the mathematical foundation to do lots of things to the beads or even to the turtle!

- change the beads
- change the turtle (stretch/shrink parts)

Why is Laplacian Related to Graphics

three dimensional (geometry processing)

You can do lots of things by changing the beads!

texture transfer

* ogre model stolen from Keenan Crane

Why is Laplacian Related to Graphics

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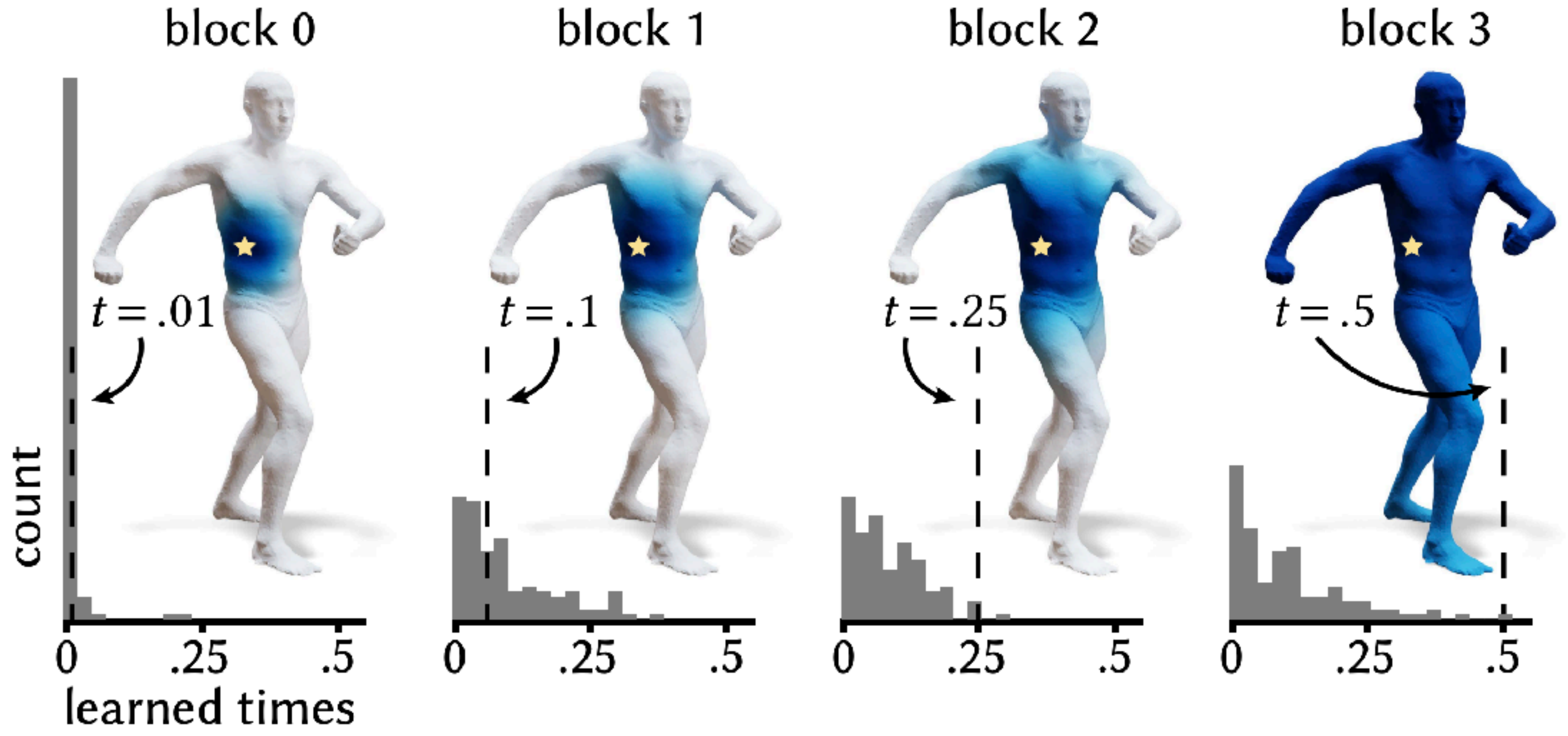


heat diffusion simulation

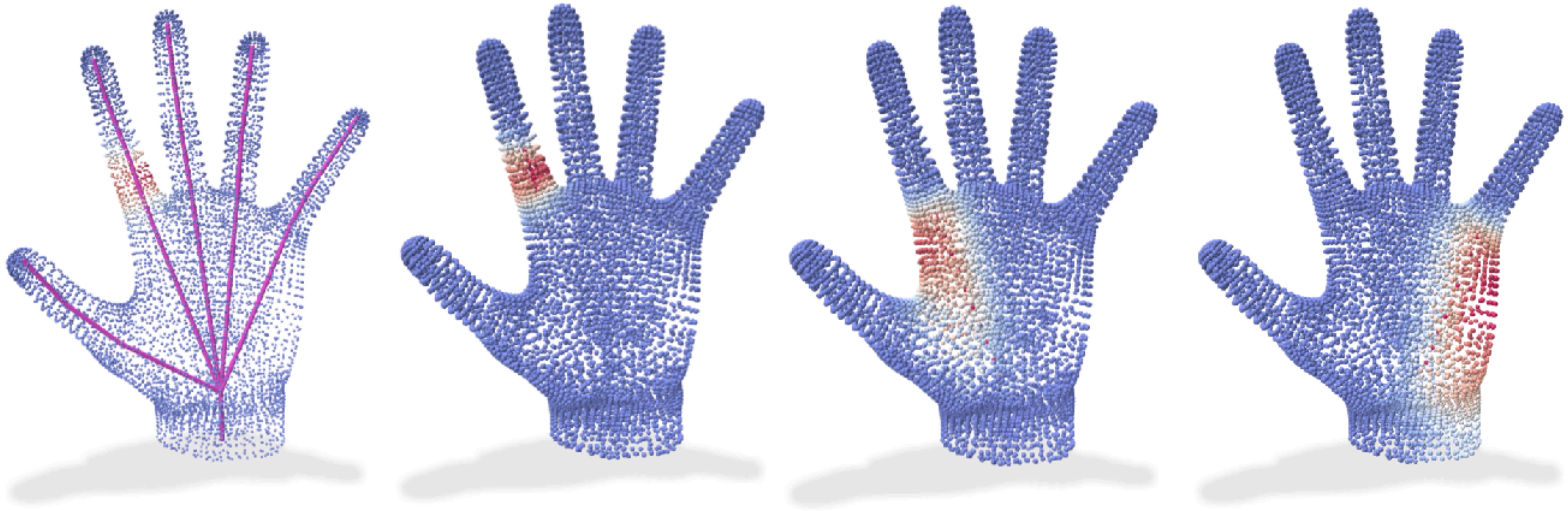
Sharp, Nicholas, et al. "Diffusionnet: Discretization agnostic learning on surfaces." *ACM Transactions on Graphics (TOG)* 41.3 (2022): 1-16.

Why is Laplacian Related to Graphics

three dimensional (geometry processing)

You can do lots of things by changing the knitting!

your beads: bounded biharmonic weights



animation (the beads)

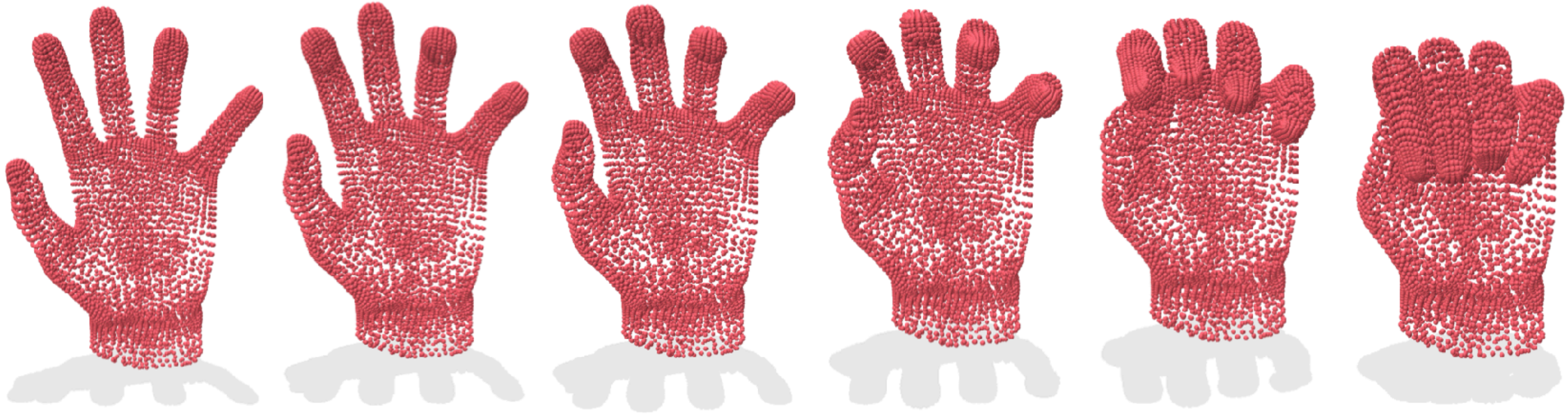
Jacobson, Alec, et al. "Bounded biharmonic weights for real-time deformation." *ACM Trans. Graph.* 30.4 (2011): 78.

Why is Laplacian Related to Graphics

three dimensional (geometry processing)

You can do lots of things by changing the knitting!

changing your knitting: animating the hand



animation (the knitting change)

Jacobson, Alec, et al. "Bounded biharmonic weights for real-time deformation." *ACM Trans. Graph.* 30.4 (2011): 78.

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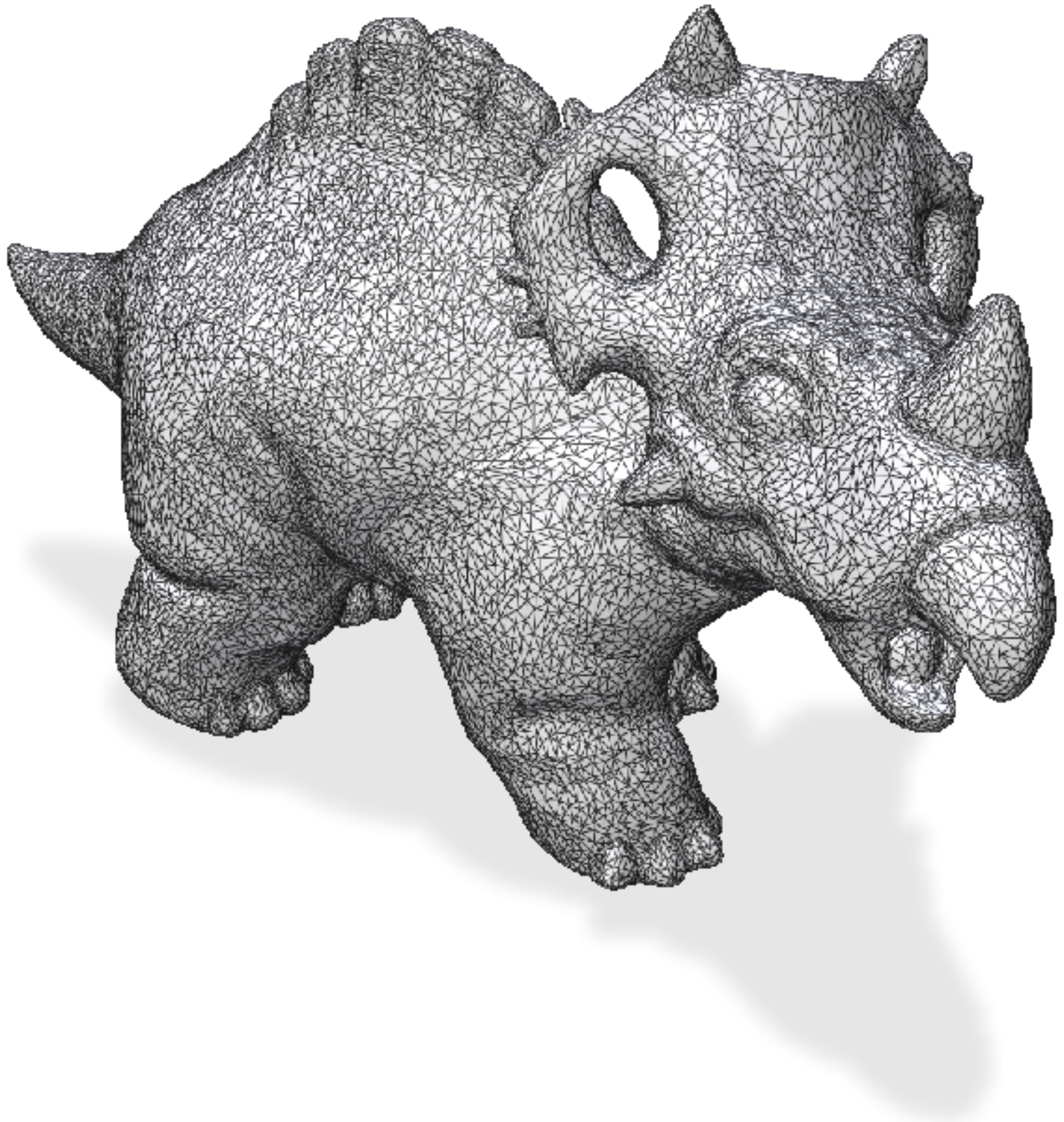

mesh simplification (the beads)

Chen, Crane He. "Estimating Discrete Total Curvature with Per Triangle Normal Variation." *SIGGRAPH Talks 2023*

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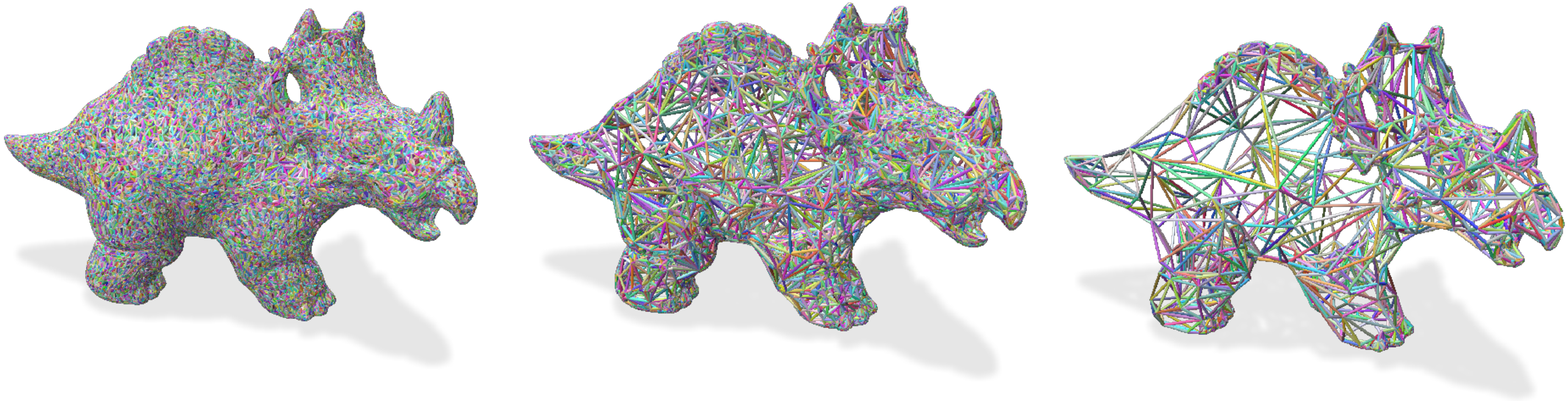

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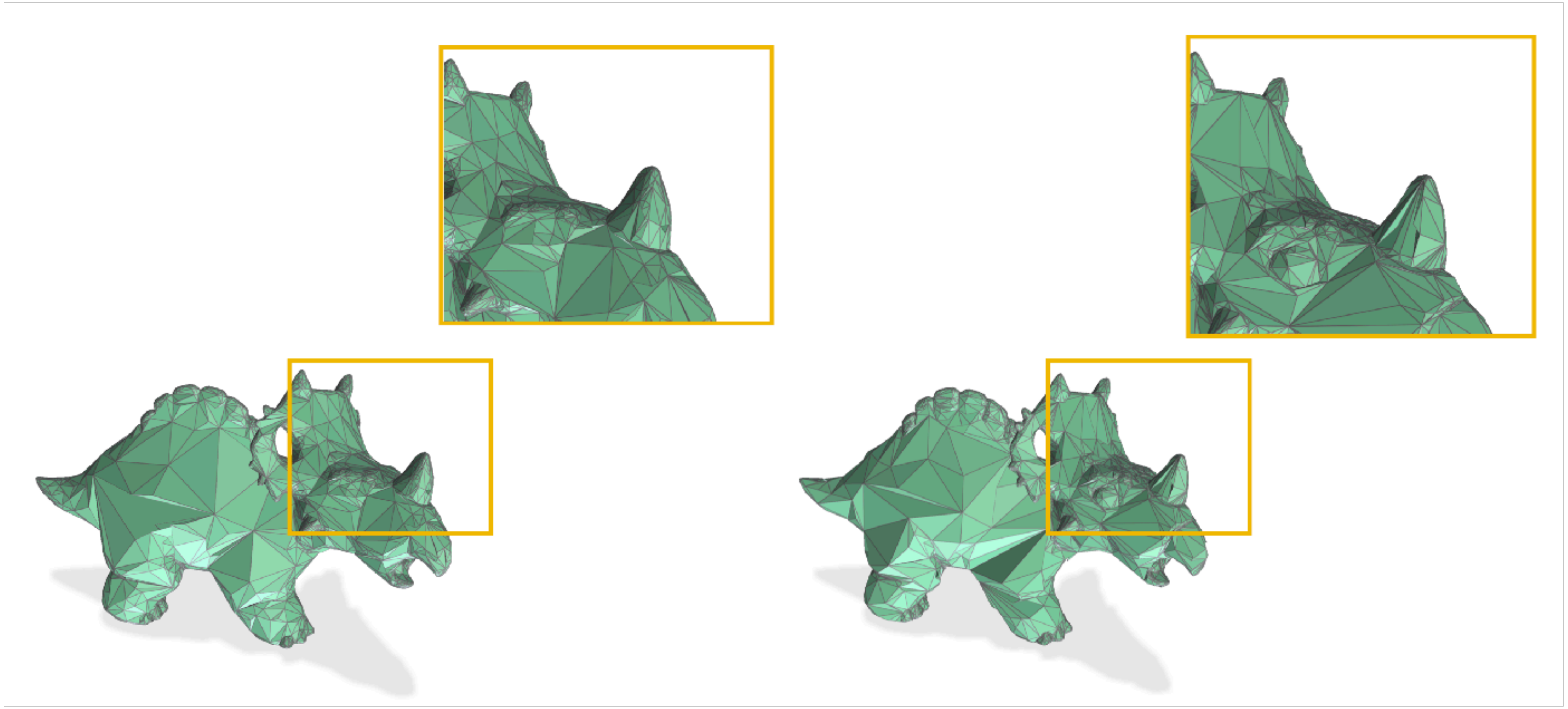

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Why is Laplacian Related to Graphics

This seemingly algebraic concept relates to geometry analysis happening in the world of computer graphics.

We call it **“Spectral Geometry”**

Laplacian is ubiquitous in computer graphics, it shows up in Dirichlet energy, Poisson equation, Laplace’s equation, wave equation, heat equation, etc.

Take-aways from Today's Lecture

- You learned intuition of the Laplacian in all dimensions
- You remembered the "beads on turtle" example
- You learned some examples of Laplacian applications

A welcome survey

https://forms.gle/XmycEdkaCN4o6gbp7

Now, your turn!

Go to the course Github page to download code!

We'll wok on coloring the bunny together!

Pair-Coding

```
int main(int argc, char *argv[])
{
  using namespace Eigen;
  using namespace std;

  // variable definition
  Eigen::MatrixXd V, PD1, PD2, PV1, PV2;
  Eigen::MatrixXi F;
  Eigen::VectorXd total_curvature, total_curvature_vis;

  // calculate total curvature
  igl::read_triangle_mesh("../data/BigBuckBunny.ply",V,F);
  igl::principal_curvature(V, F, PD1, PD2, PV1, PV2);
  total_curvature = PV1.array().square() + PV2.array().square();
  total_curvature_vis = total_curvature.array().pow(0.01);

  // visualization
  polyscope::init();
  polyscope::options::groundPlaneMode = polyscope::GroundPlaneMode::ShadowOnly;
  auto psMesh = polyscope::registerSurfaceMesh("bunny", V, F);
  auto TotalCurvature = polyscope::getSurfaceMesh("bunny");
  auto ScalarQuantity1 = TotalCurvature->addVertexScalarQuantity("TotalCurvature", total_curvature_vis);
  ScalarQuantity1->setColorMap("jet");
  ScalarQuantity1->setEnabled(true);
  polyscope::options::shadowDarkness = 0.1;
  polyscope::show();

}
```

https://github.com/HeCraneChen/500.111.40_exploring_laplacian_graphics_course/tree/main/week3

Are There Any Questions?