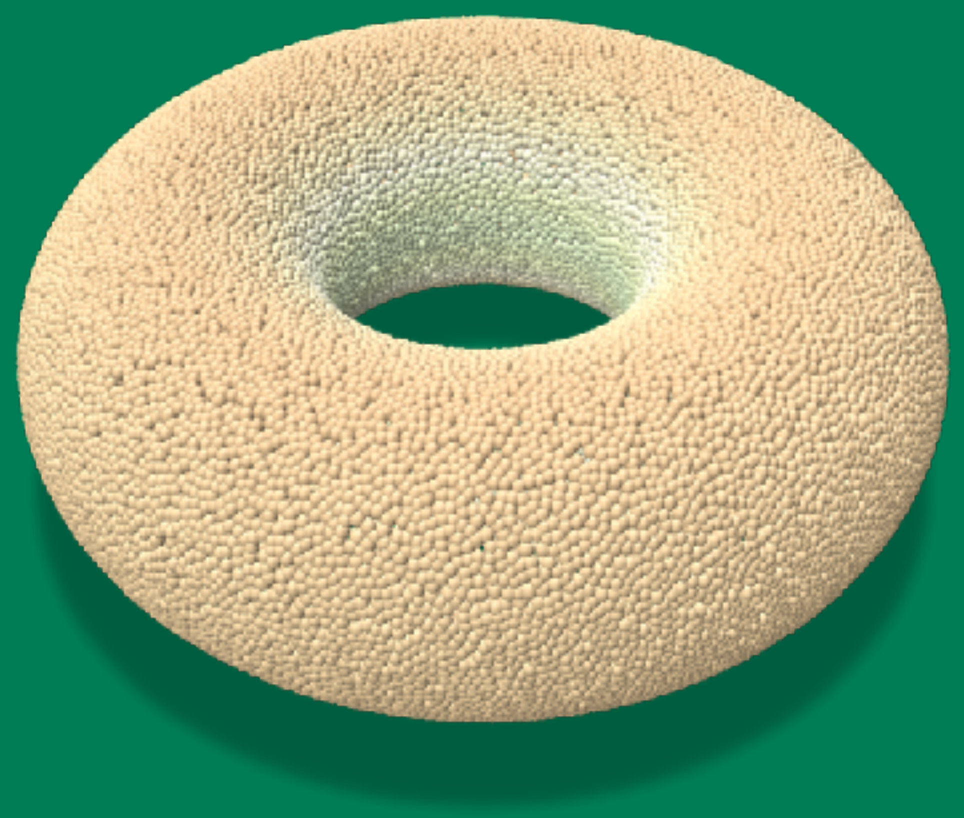
Exploring the Laplacian in Computer Graphics

Week 5



Crane He Chen

The Johns Hopkins University

2023 Fall

Review Questions

What examples can you think of for Laplacian in 1D, 2D, 3D?

From Last Week

one dimensional (geometry processing)

You have a pearl necklace
with pearls of graduated sizes

$f(t) =$ **size of pearl t**

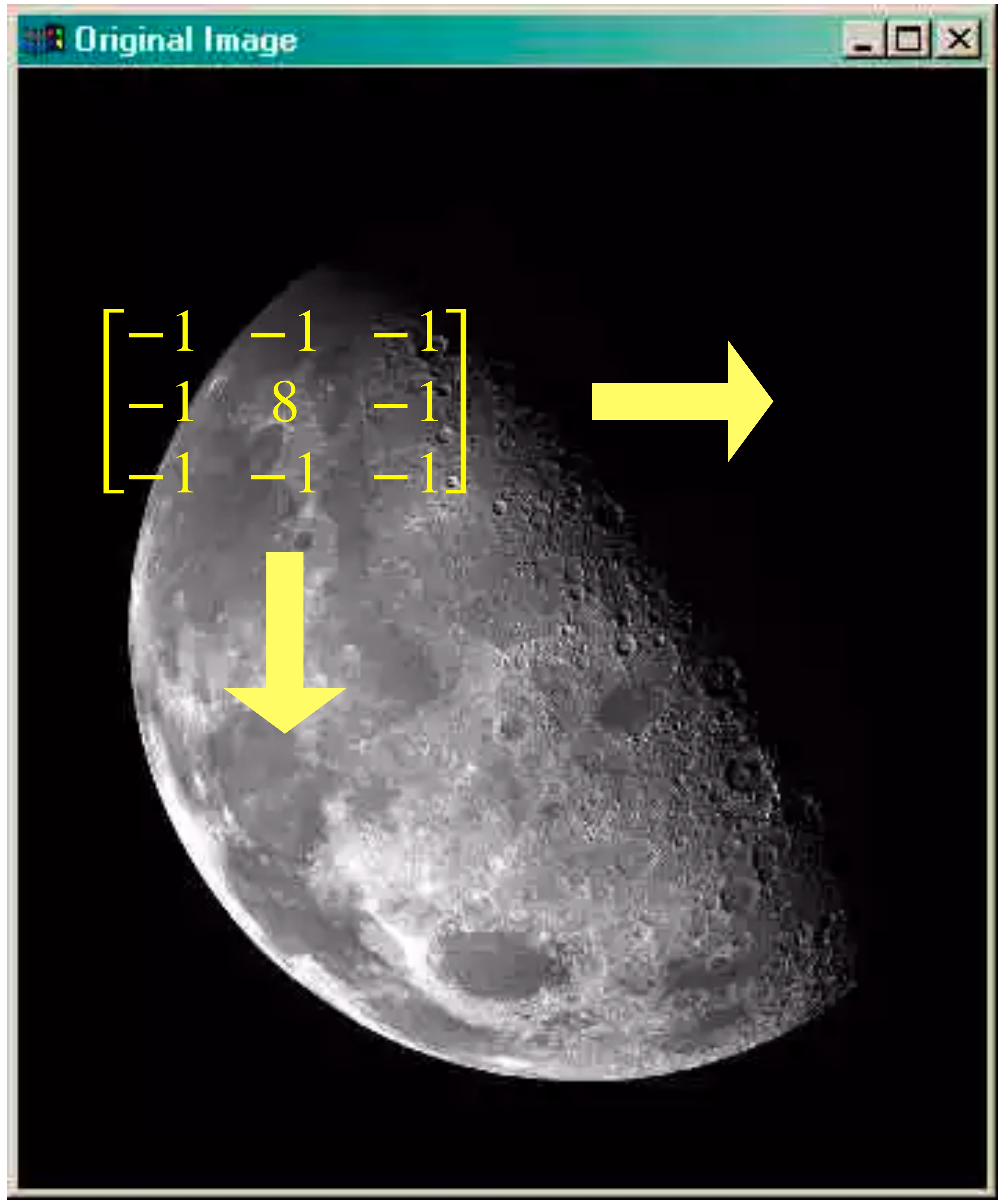
$$\Delta f[t] = f[t+1] - 2f[t] + f[t-1]$$

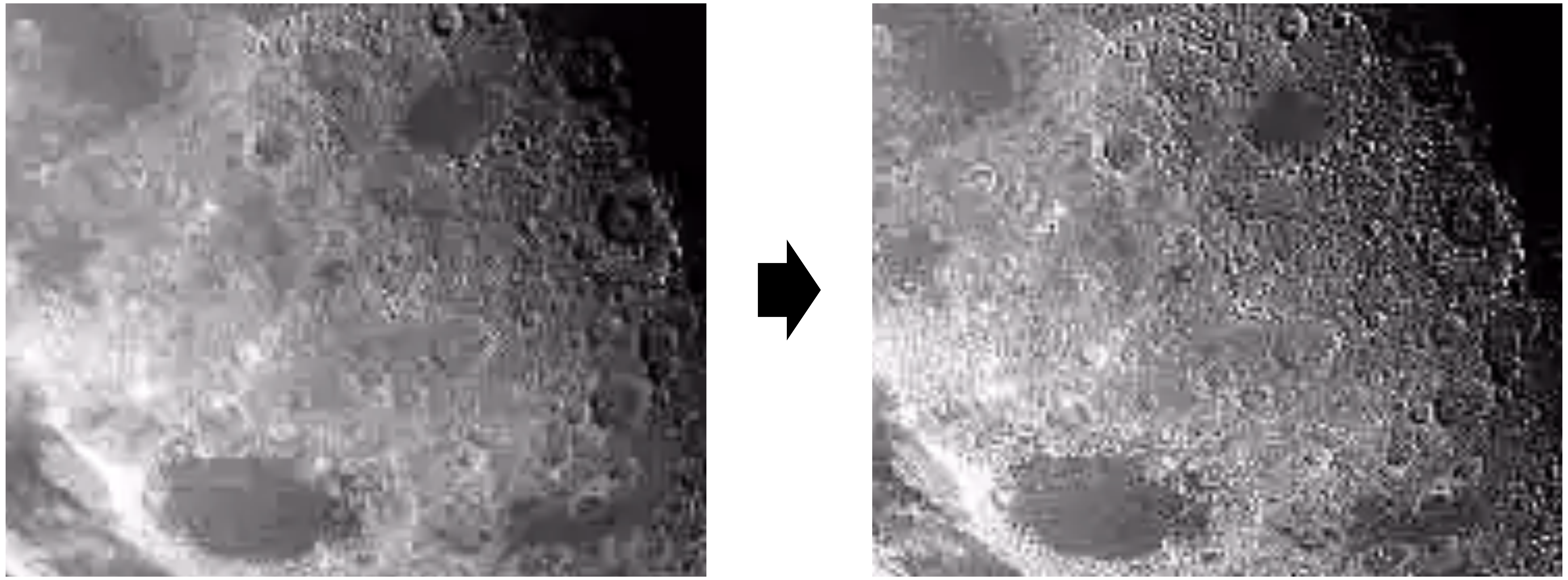
Q: When would $|\Delta f[t]|$ be large?

A: when the size of pearl t is
significantly from its neighbors!

From Last Week

two dimensional (image processing)





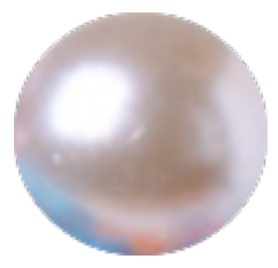
From Last Week

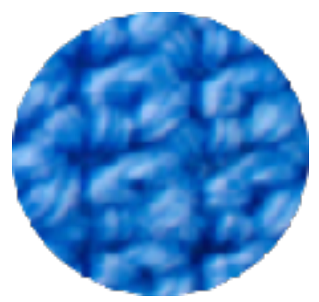
three dimensional (geometry processing)

scalar function living on vertices: beads

Laplacian: helps us understand how each bead is different from its neighboring beads

"quality inspector"
of beads distribution





beads
(the scalar function)

"bridge"

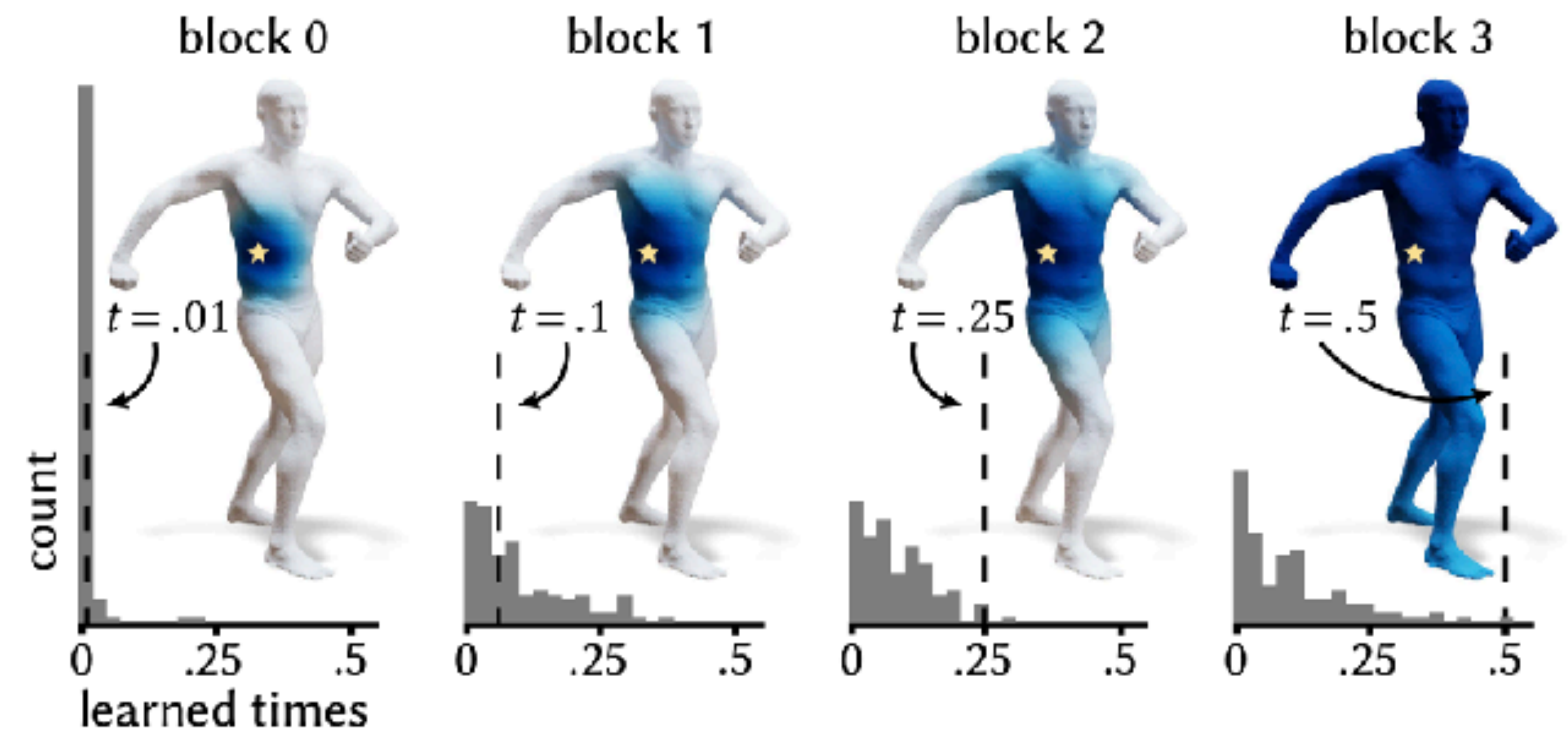
knit
(the shape)

From Last Week

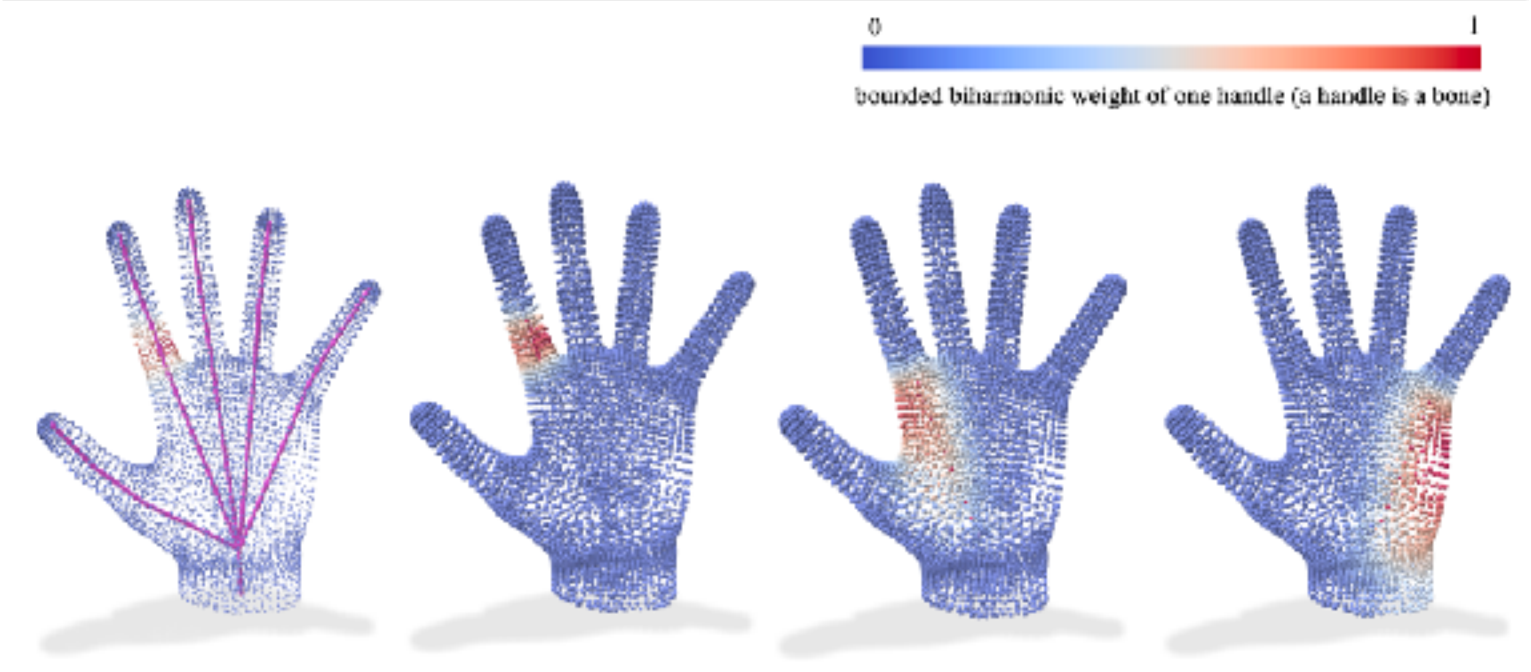
three dimensional (geometry processing)

You can do lots of things by changing the beads!

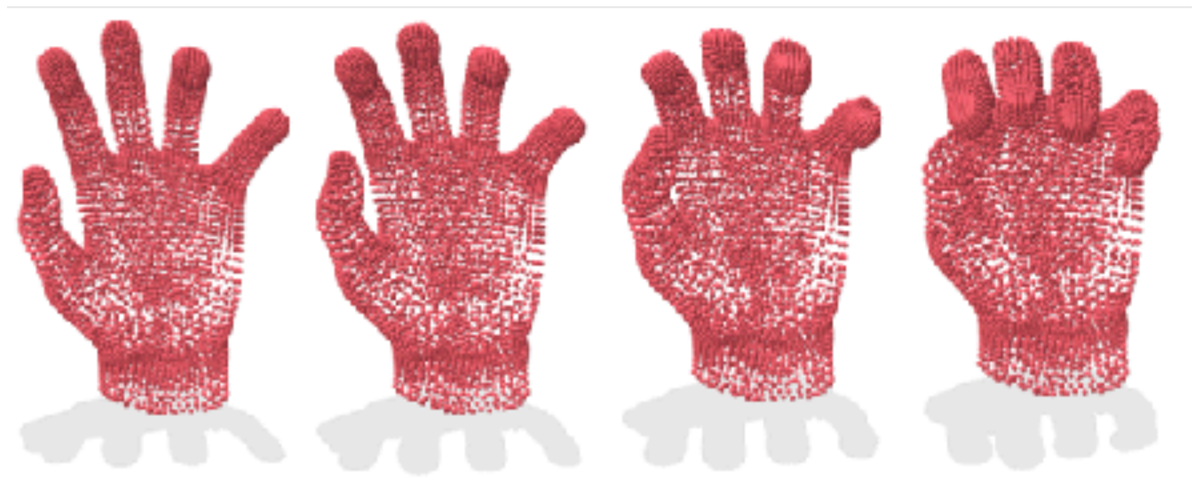
You can do lots of things by changing the knitting!



heat diffusion simulation

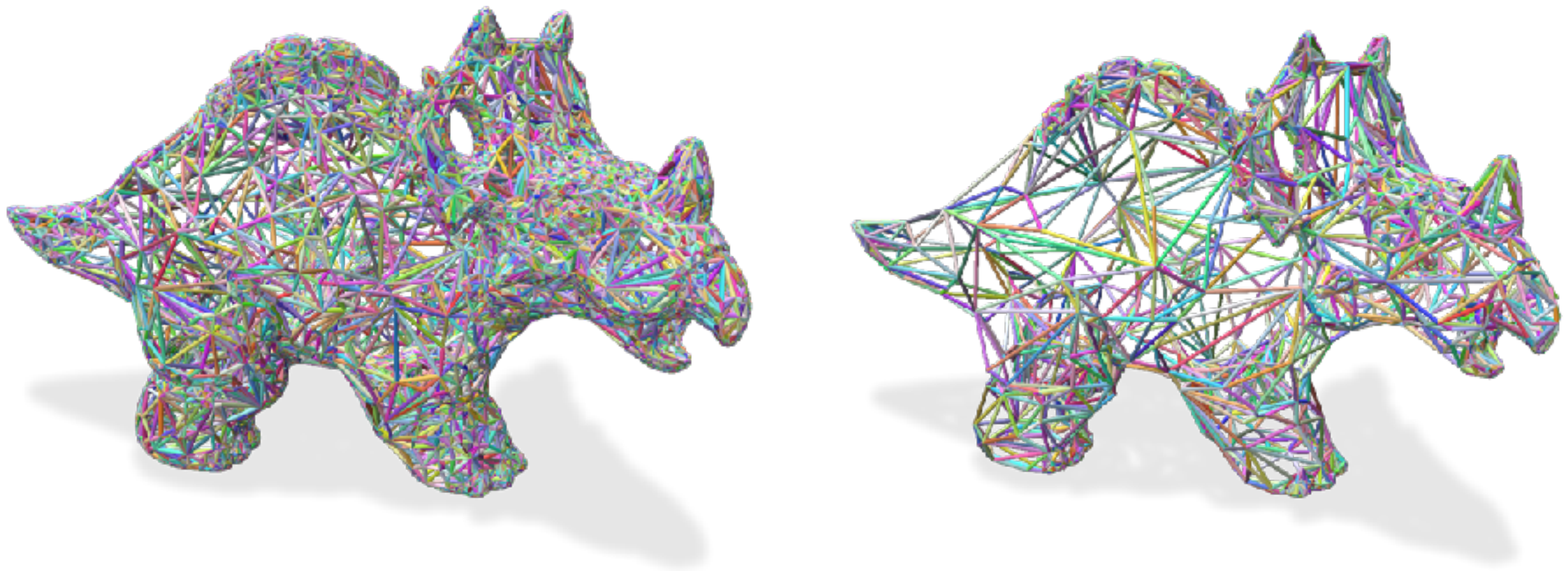


animation (the beads)



animation (the knitting change)

mesh simplification (the beads)



mesh simplification (the knitting change)

Sharp, Nicholas, et al. "Diffusionnet: Discretization agnostic learning on surfaces." *ACM Transactions on Graphics (TOG)* 41.3 (2022): 1-16.
Jacobson, Alec, et al. "Bounded biharmonic weights for real-time deformation." *ACM Trans. Graph.* 30.4 (2011): 78.
Chen, Crane He. "Estimating Discrete Total Curvature with Per Triangle Normal Variation." *SIGGRAPH Talks 2023*

Review Questions

Come to the blackboard!

Where does this function live?

$$f(x, y, z) = x^2 + y^2 + z^2$$

Review Questions

What is the Laplacian?

Let’s take turns, I want to hear thoughts from everyone!

Review Questions

What is “mesh”?

Let’s take turns, I want to hear thoughts from everyone!

Many Definitions

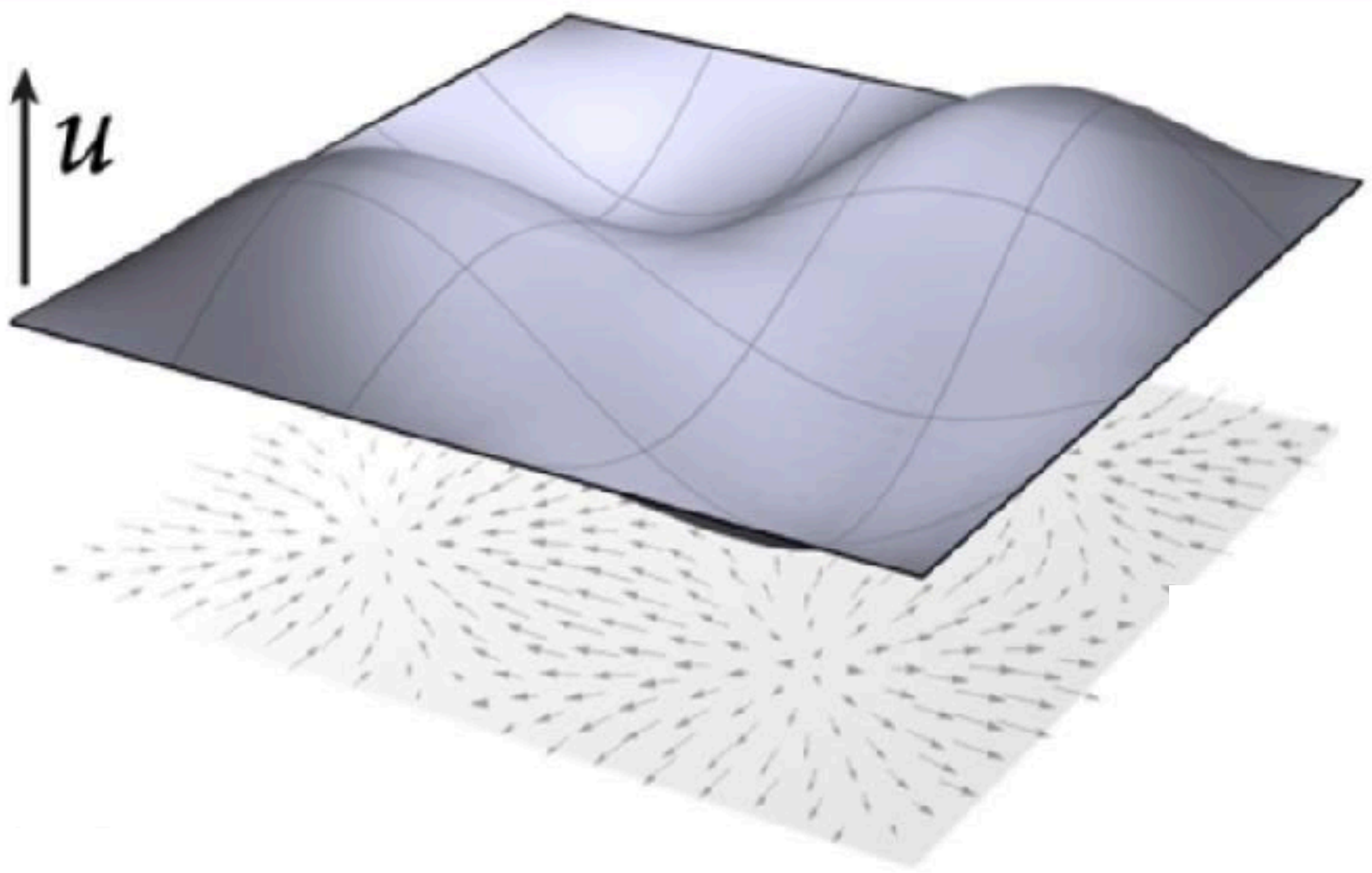
- Deviation from local average
- Sum of second derivatives
- Divergence of gradient

∇ pronounced as "nabla"

Δ pronounced as "delta"

Math Preliminaries

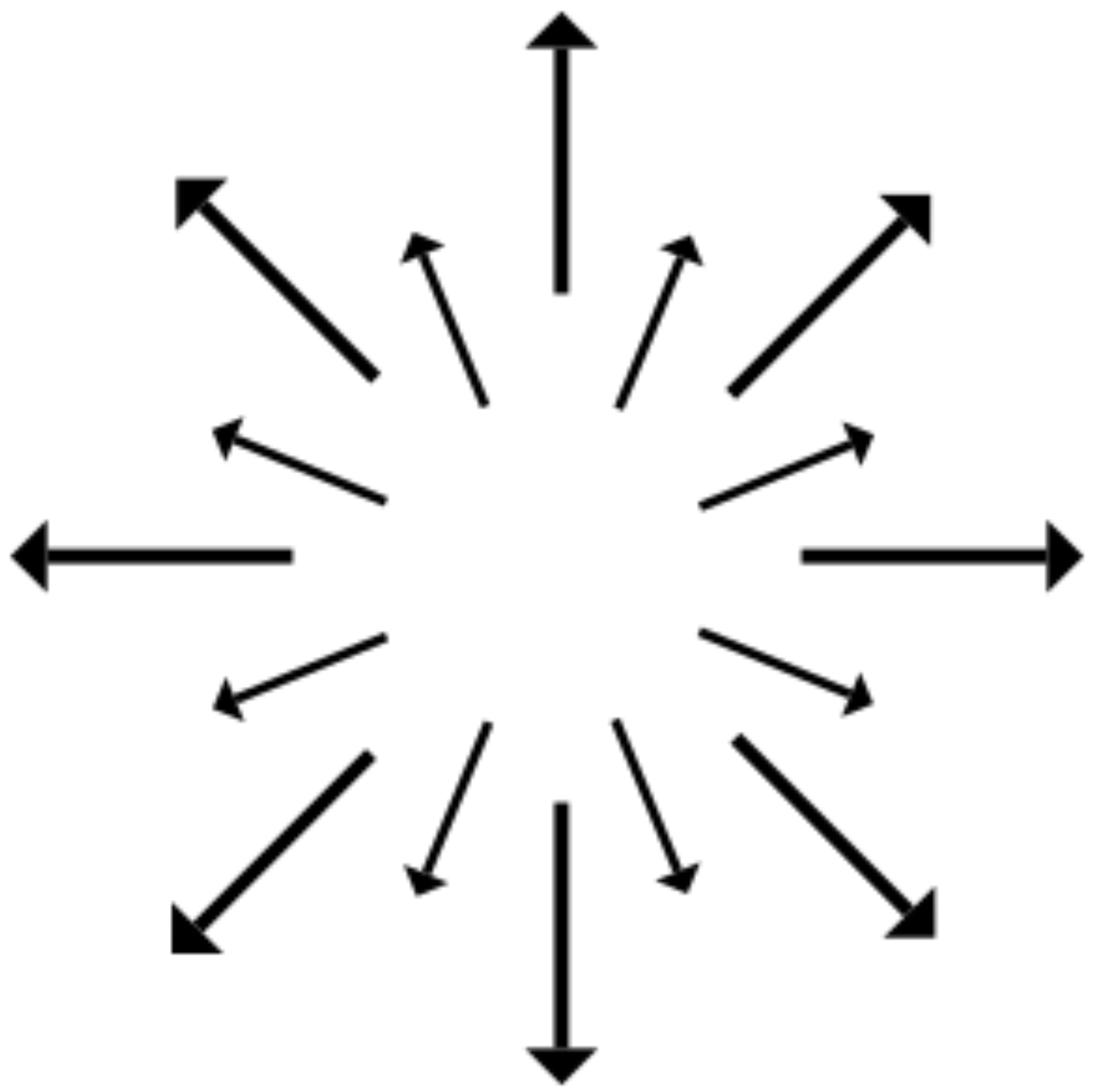
∇ gradient



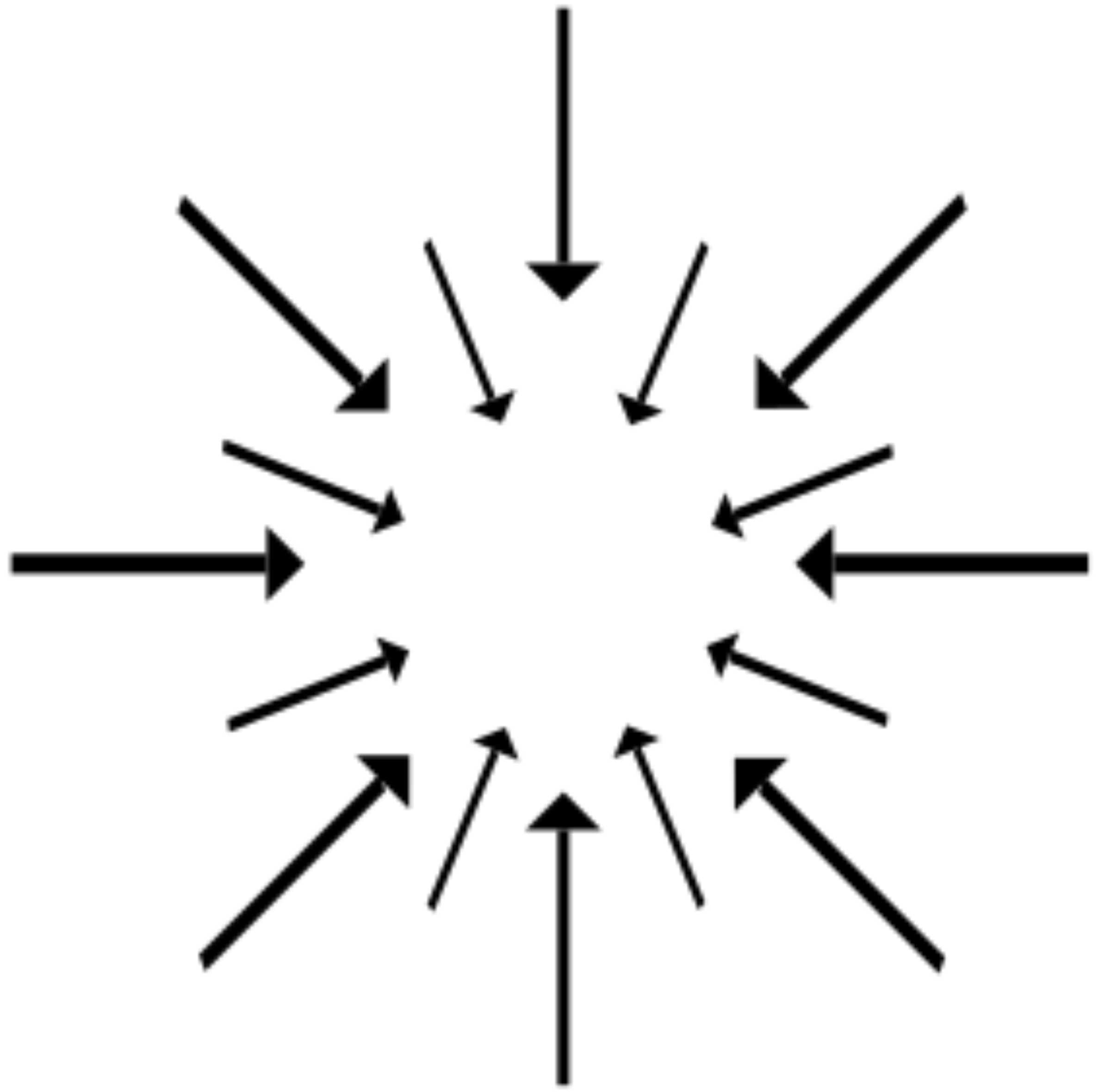
This is not to be confused with Δ Laplacian

Math Preliminaries

$\nabla \cdot$ divergence



divergence > 0
intuition: “source”



divergence < 0
intuition: “sink”

$$\int_M f(x)dx$$ integral

Think of it as "sum", continuously

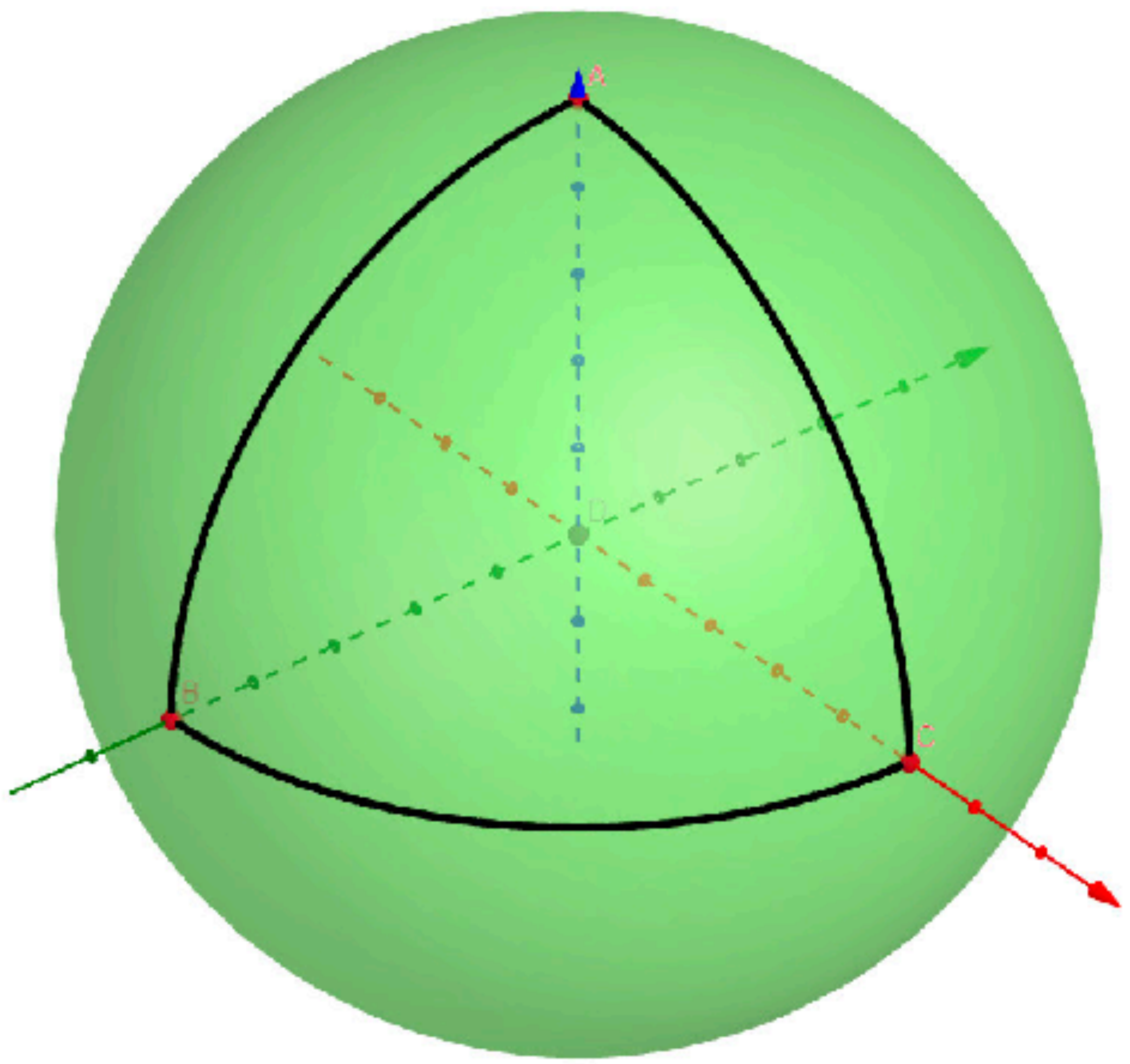
M means the region you want the "sum" to happen. Let's say M means 1~5

$$sum \approx f(1) + f(2) + f(3) + \ldots$$

$$sum \approx f(1) + f(1.1) + f(1.2) + \ldots + f(3.1) + f(3.2) + \ldots$$

Let’s formalize this a bit more!

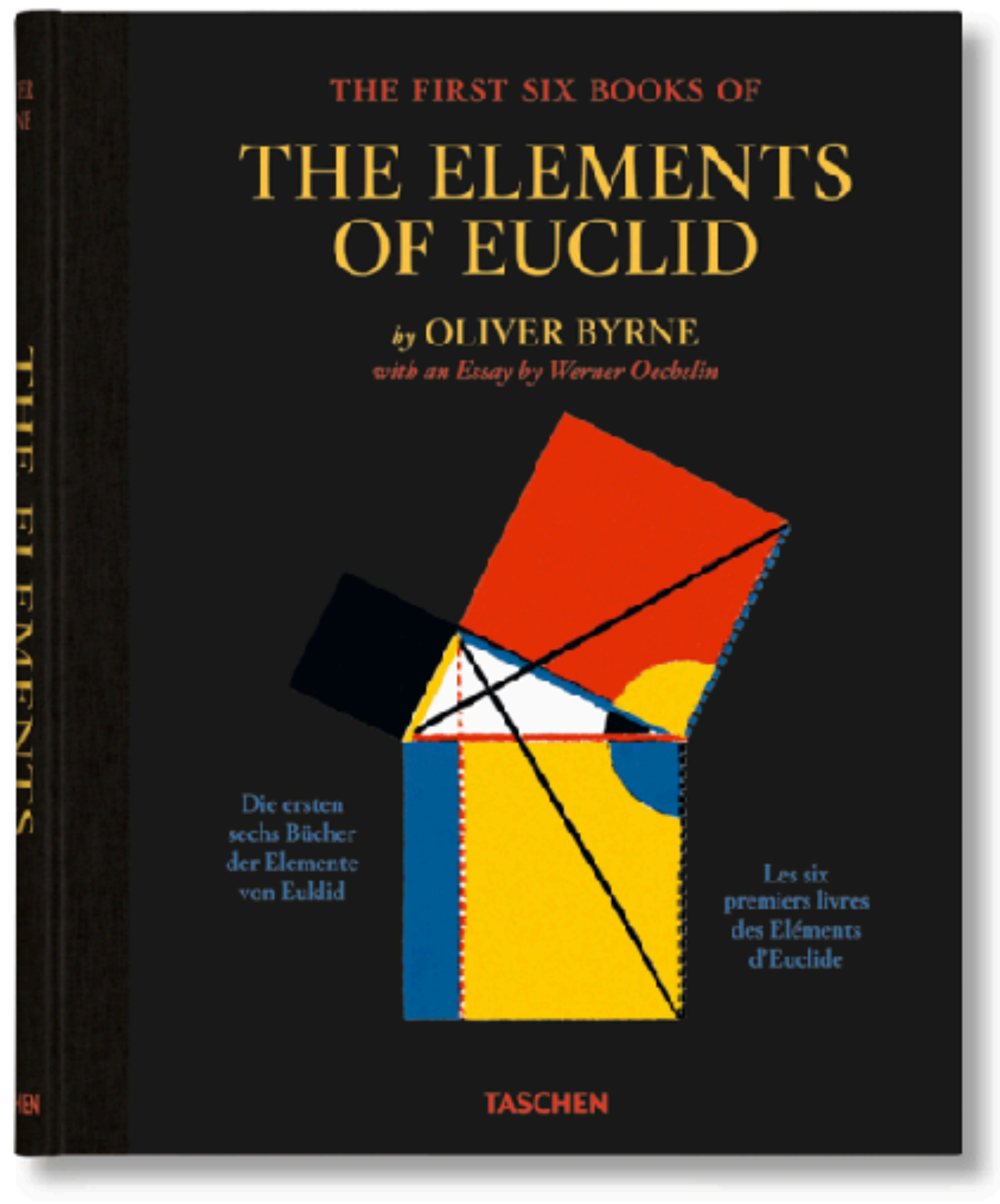
What’s the sum of angles of a triangle on a sphere?



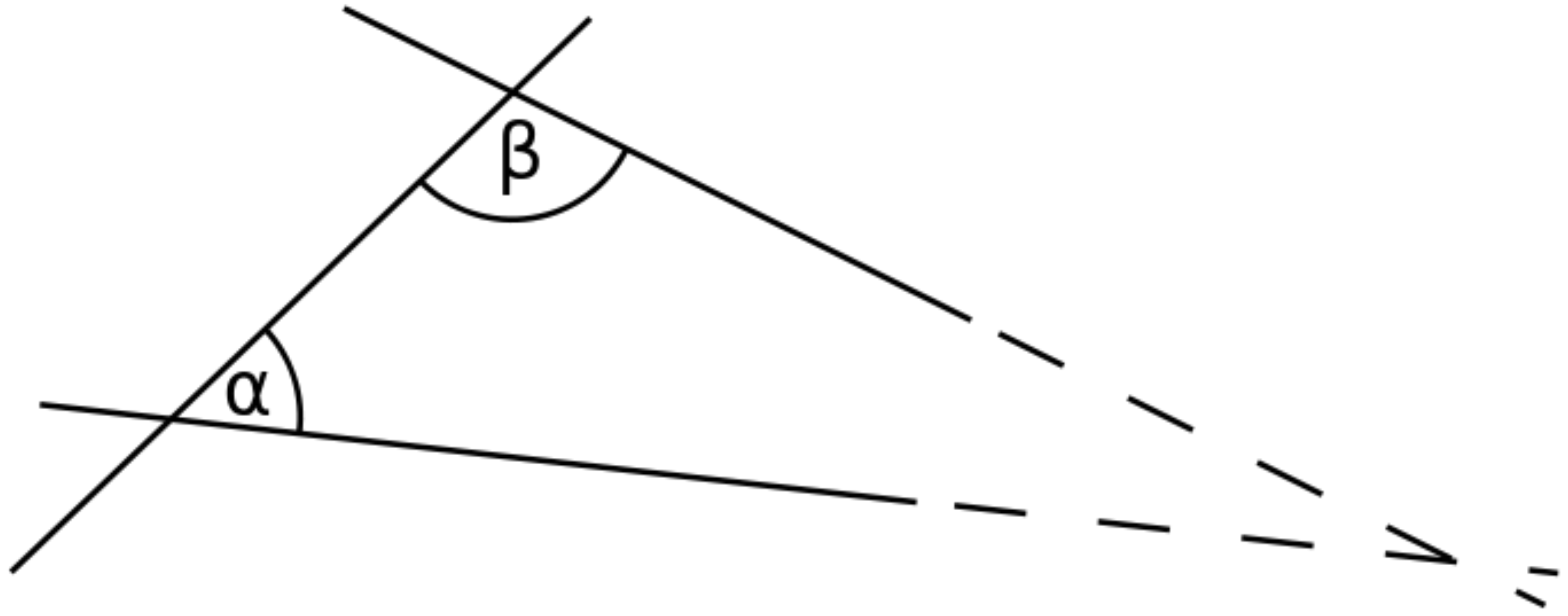
https://en.wikipedia.org/wiki/Laplace%E2%80%93Beltrami_operator

Let’s formalize this a bit more!

High school: Euclidean space



Euclid’s postulate 5:



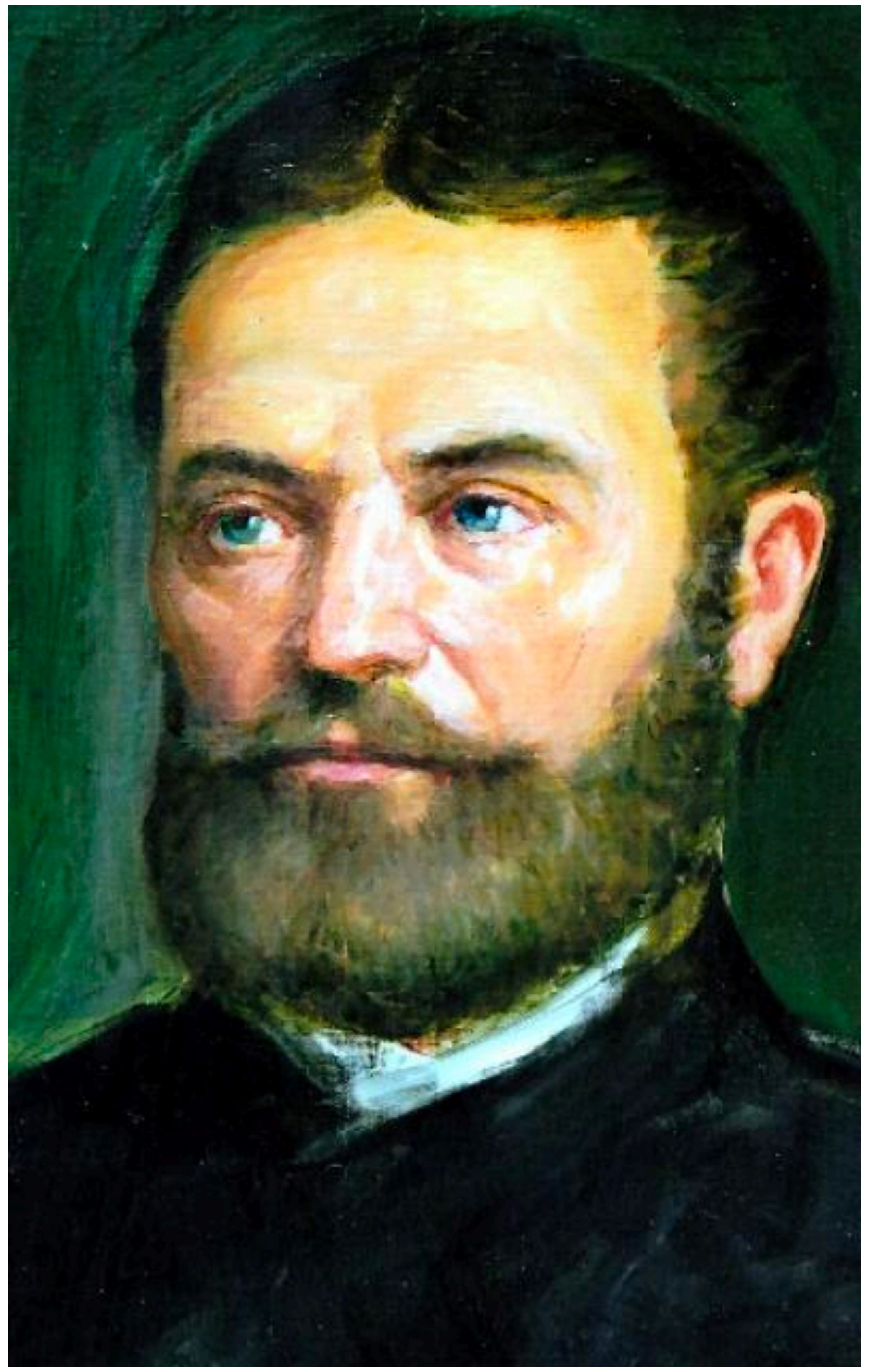
When two straight lines intersect with a line segment, if the sum of the interior angles alpha and beta is less than 180, the two straight lines meet on that side.

Formalizing Laplacian (Geometry)

Let’s formalize this a bit more!

High school: Euclidean space

When the German geometer János Bolyai tried to prove the parallel postulate, he dis-proved it instead. And that led to an important discovery called Non-Euclidean geometry, which was only recognized after he passed away.



János Bolyai

https://en.wikipedia.org/wiki/Euclidean_space https://en.wikipedia.org/wiki/Parallel_postulate

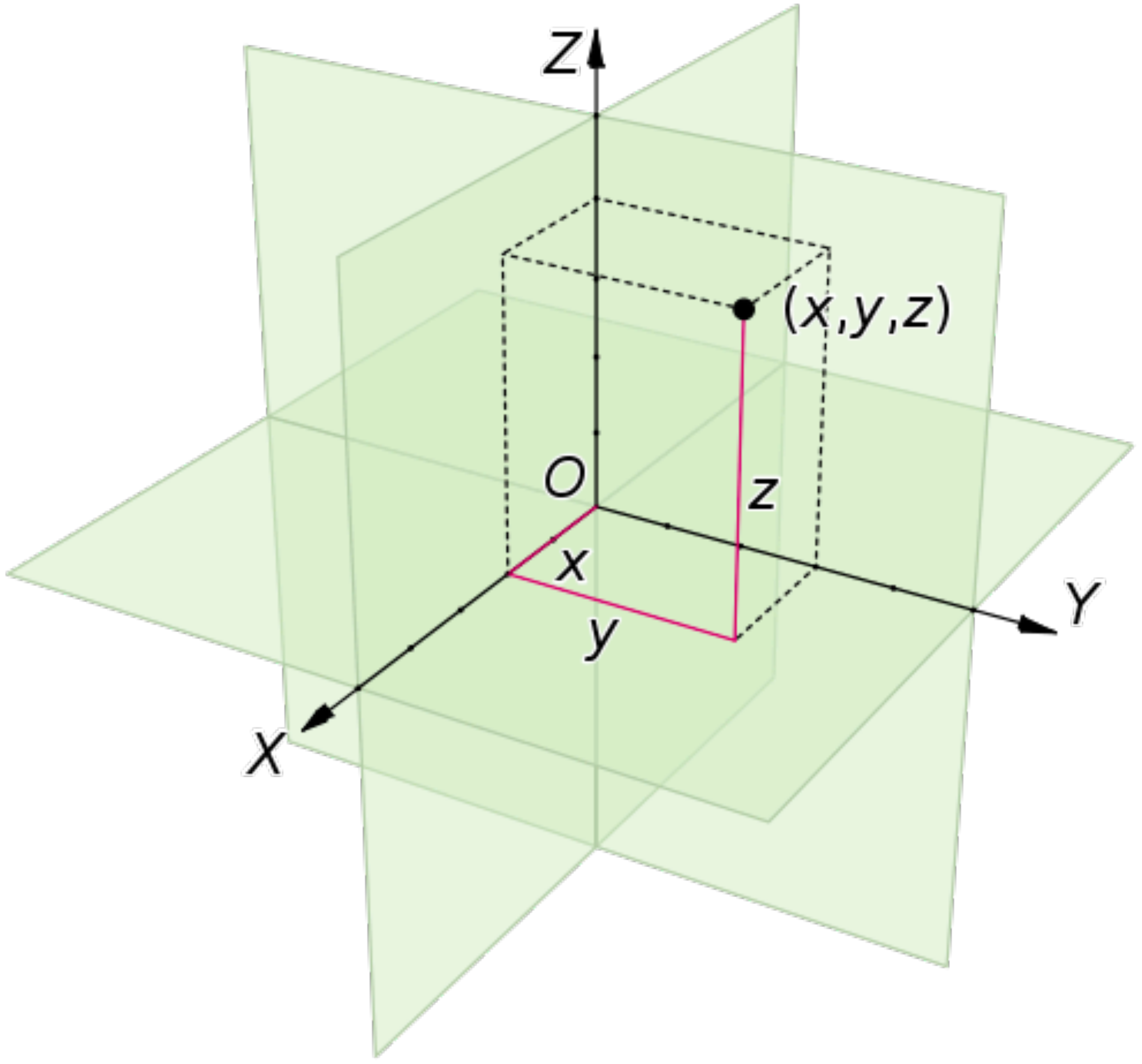
Formalizing Laplacian (Geometry)

Let's formalize this a bit more!

Δ

"The Laplacian"

(Euclidean domain)



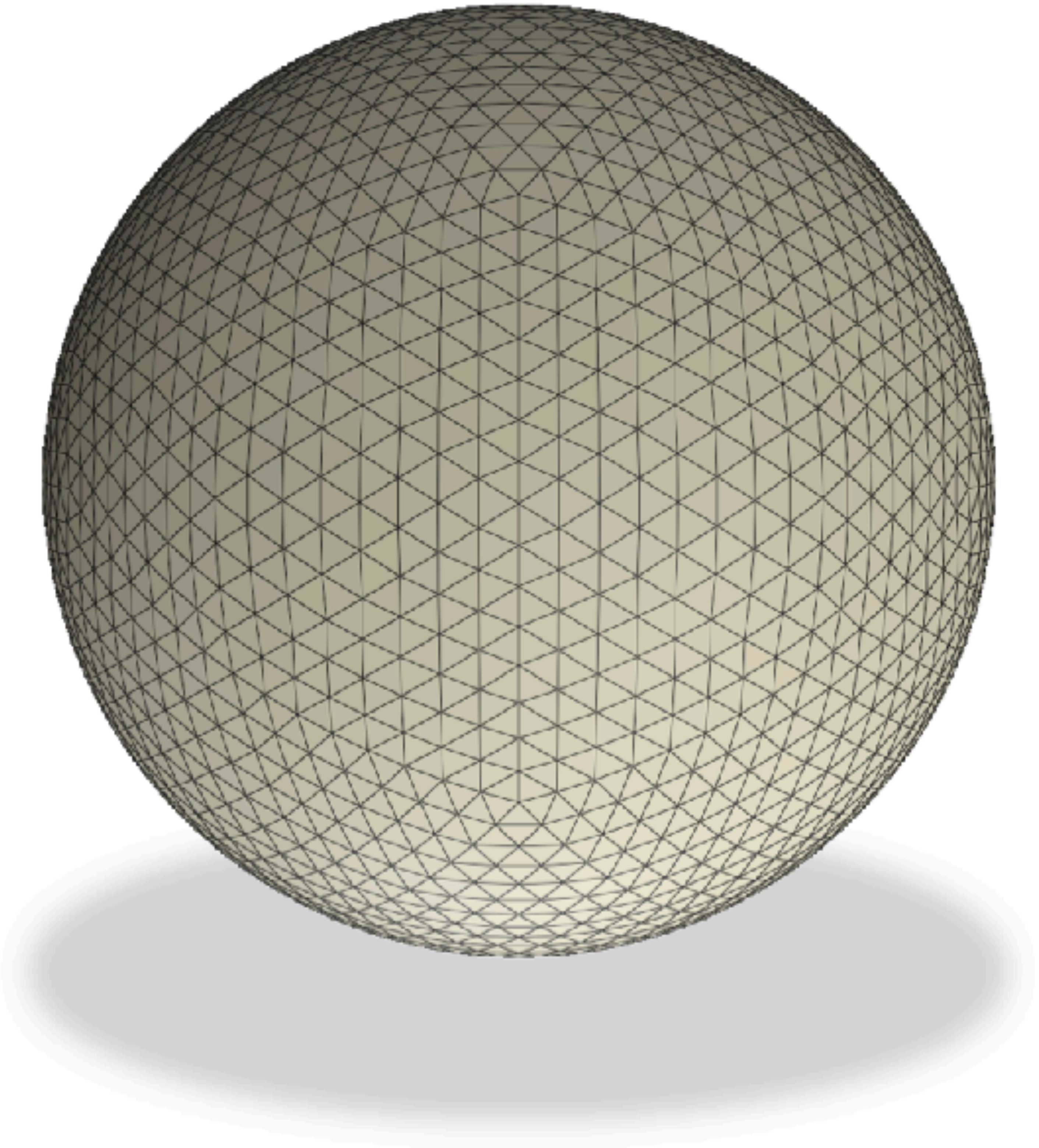
https://en.wikipedia.org/wiki/Euclidean_space

Formalizing Laplacian (Geometry)

Let's formalize this a bit more!

Δ **"Laplace-Beltrami Operator"**

(curved domain)

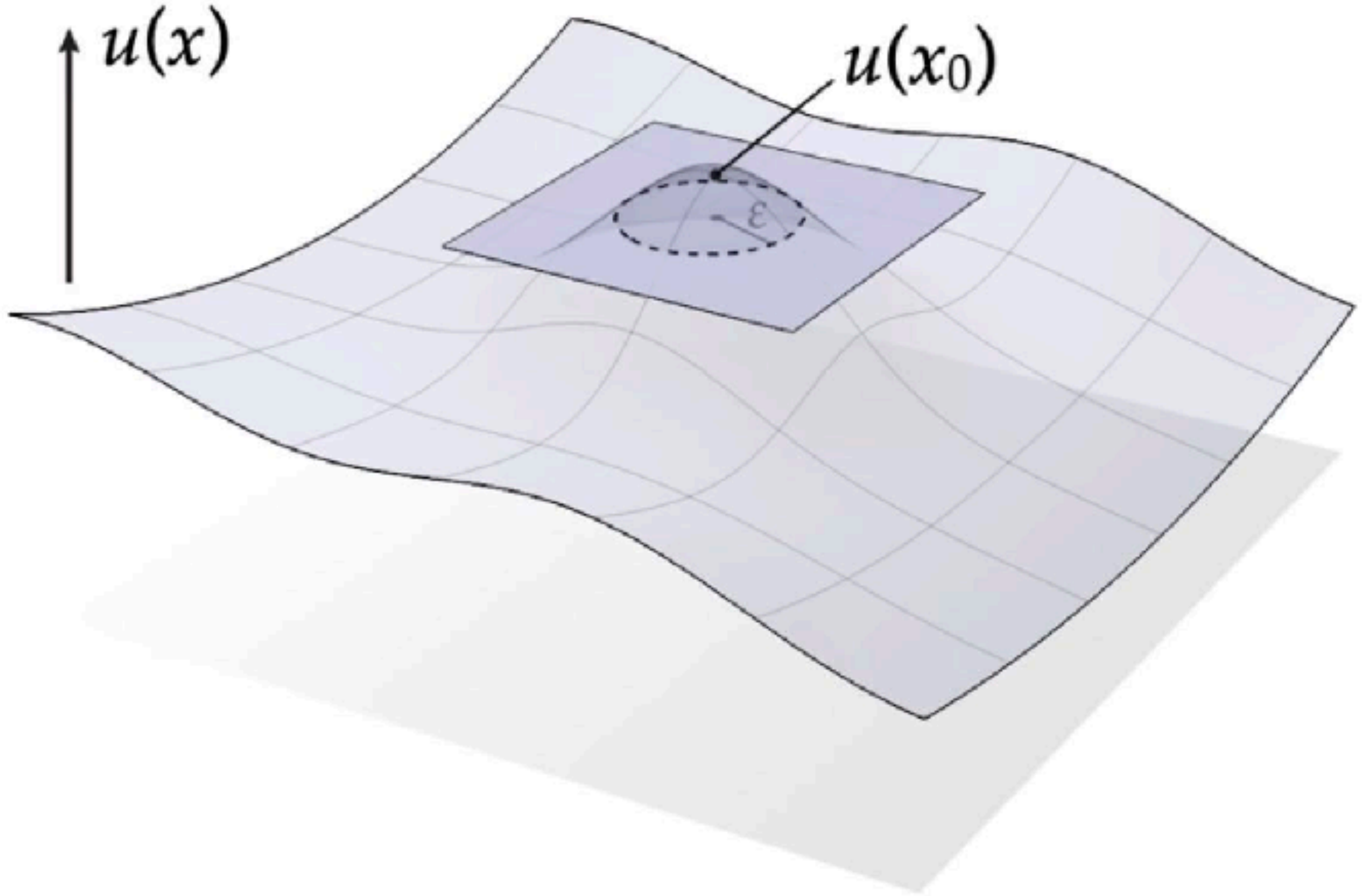


https://en.wikipedia.org/wiki/Laplace%E2%80%93Beltrami_operator

Formalizing Laplacian (Geometry)

Let’s formalize this a bit more!

Key idea: Laplacian is deviation from local average



deviation from local average

Figure stolen from Keenan Crane slides

Formalizing Laplacian (Derivative)

Let's formalize this a bit more!

$$\Delta$$

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

sum of second derivatives

Who wants to try this? What's Δu ?

$$u(x_1, x_2, x_3) = x_1^3 + x_2^3 + x_3^3$$

$$\Delta u = 6x_1 + 6x_2 + 6x_3$$

Formalizing Laplacian (Calculus)

Let's formalize this a bit more!

Δ "divergence of gradient"

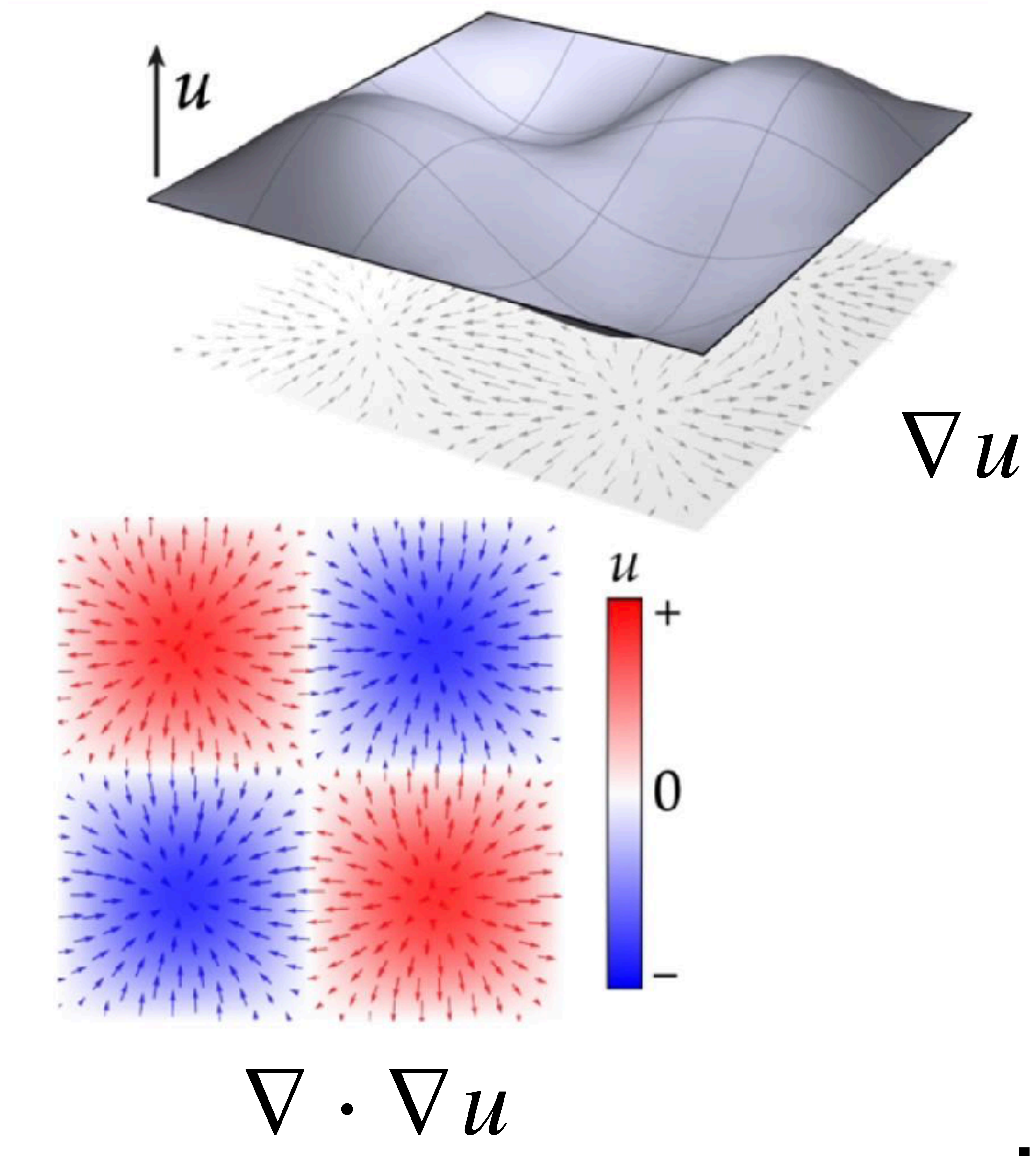
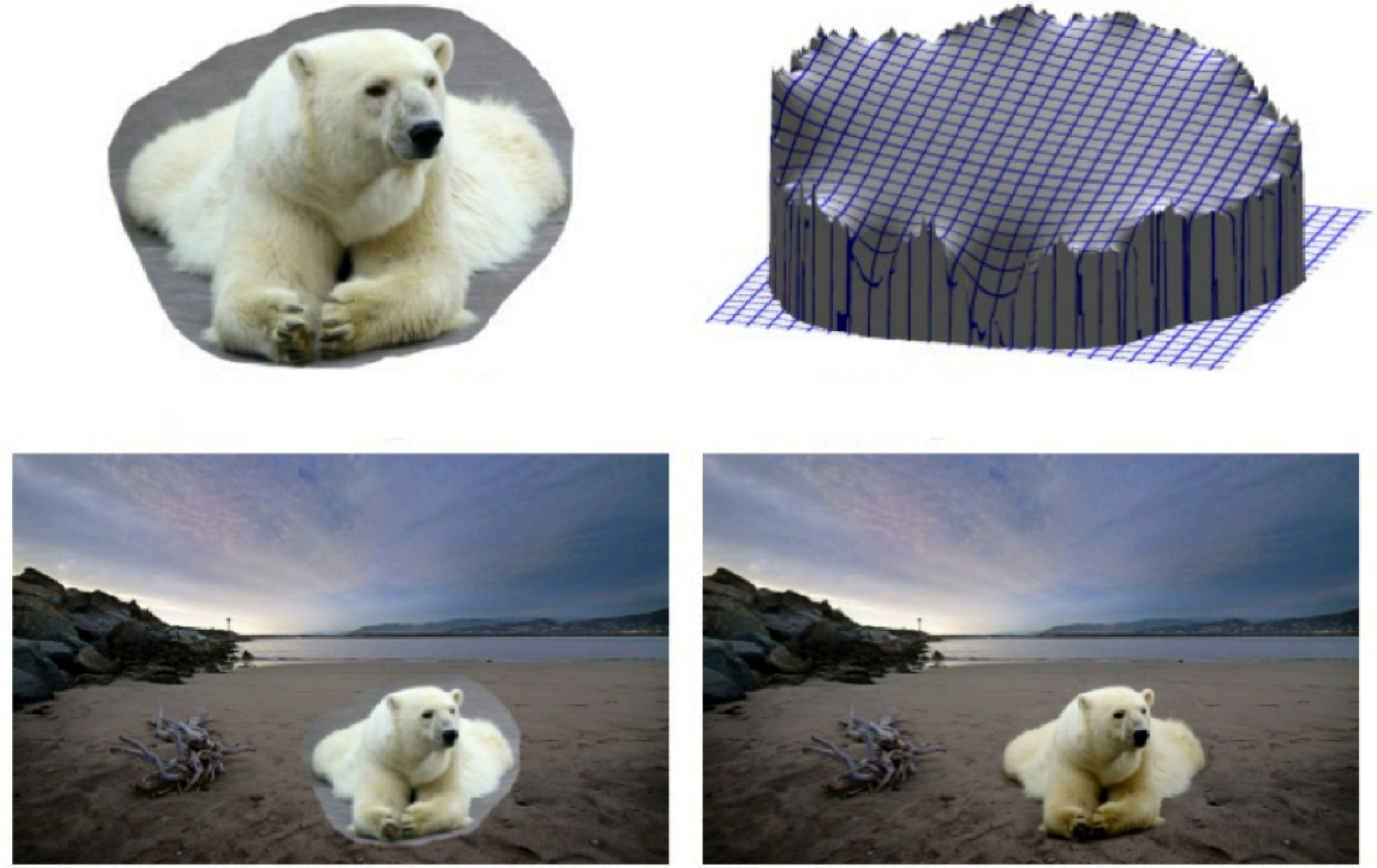


Figure stolen from Keenan Crane slides

Dirichlet Energy

Dirichlet energy is closely related to the Laplacian!



Farbman *et al.* SIGGRAPH 2009

Figure stolen from Misha Kazhdan slides

Dirichlet Energy

Dirichlet energy is closely related to the Laplacian!

Given boundary, can you find a function that fills in the interior “as smooth as possible”?

Minimize the Dirichlet energy!

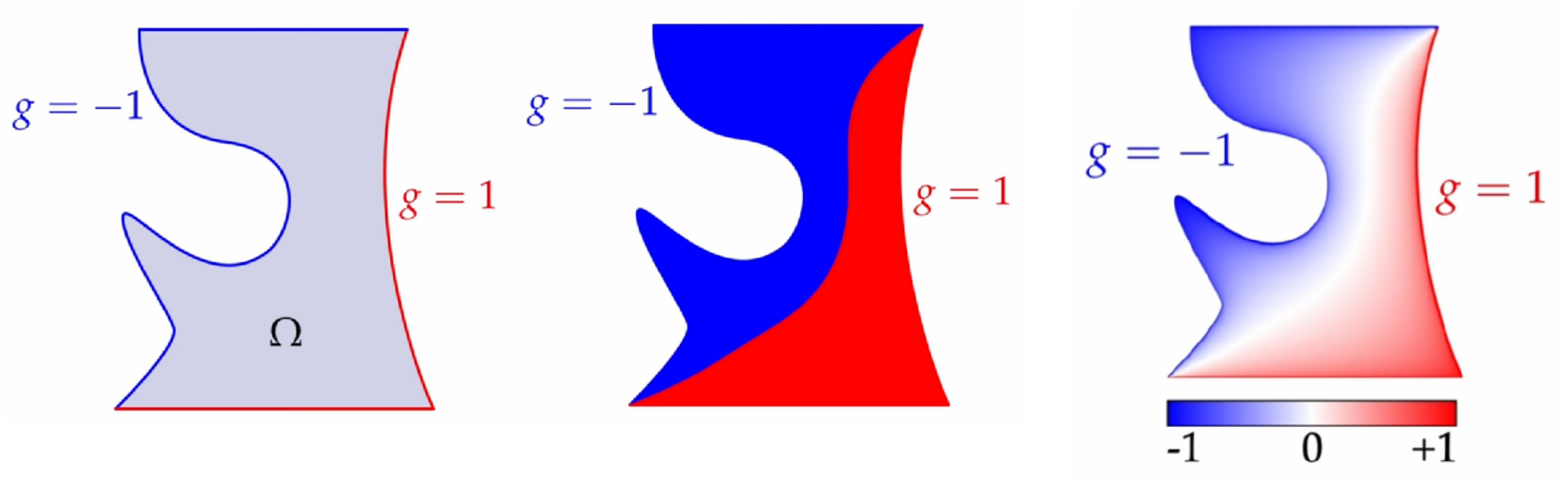


Figure stolen from Keenan Crane slides

Dirichlet Energy

Dirichlet energy is closely related to the Laplacian!

Minimizing the Dirichlet energy is equal to
solving a Laplace equation

$$\int_N \|\nabla h(p)\|^2 dp \iff \Delta h = 0$$

Take-aways from Today's Lecture

- You learned what's beyond Euclidean geometry
- You learned a terminology "Laplace-Beltrami Operator"
- You learned three formal definitions of the Laplacian
- You learned a terminology "Dirichlet Energy"

Pair-Coding

So far, you are coding:

- How to visualize a 3D data using python
- How to use popular libraries in computer graphics, Libigl, Polyscope
- How to compile and run the first algorithm in C++ using Make

Pair-Coding

More gears for art contest!

swept volume

cubic stylization

https://www.dgp.toronto.edu/projects/swept-volumes/
https://www.dgp.toronto.edu/projects/cubic-stylization/

Now, your turn!

Go to the course Github page to download code!

We'll wok on coloring the bunny together!

Pair-Coding

```
int main(int argc, char *argv[])
{
  using namespace Eigen;
  using namespace std;

  // variable definition
  Eigen::MatrixXd V, PD1, PD2, PV1, PV2;
  Eigen::MatrixXi F;
  Eigen::VectorXd total_curvature, total_curvature_vis;

  // calculate total curvature
  igl::read_triangle_mesh("../data/BigBuckBunny.ply",V,F);
  igl::principal_curvature(V, F, PD1, PD2, PV1, PV2);
  total_curvature = PV1.array().square() + PV2.array().square();
  total_curvature_vis = total_curvature.array().pow(0.01);

  // visualization
  polyscope::init();
  polyscope::options::groundPlaneMode = polyscope::GroundPlaneMode::ShadowOnly;
  auto psMesh = polyscope::registerSurfaceMesh("bunny", V, F);
  auto TotalCurvature = polyscope::getSurfaceMesh("bunny");
  auto ScalarQuantity1 = TotalCurvature->addVertexScalarQuantity("TotalCurvature", total_curvature_vis);
  ScalarQuantity1->setColorMap("jet");
  ScalarQuantity1->setEnabled(true);
  polyscope::options::shadowDarkness = 0.1;
  polyscope::show();

}
```

https://github.com/HeCraneChen/500.111.40_exploring_laplacian_graphics_course/tree/main/week3

Are There Any Questions?