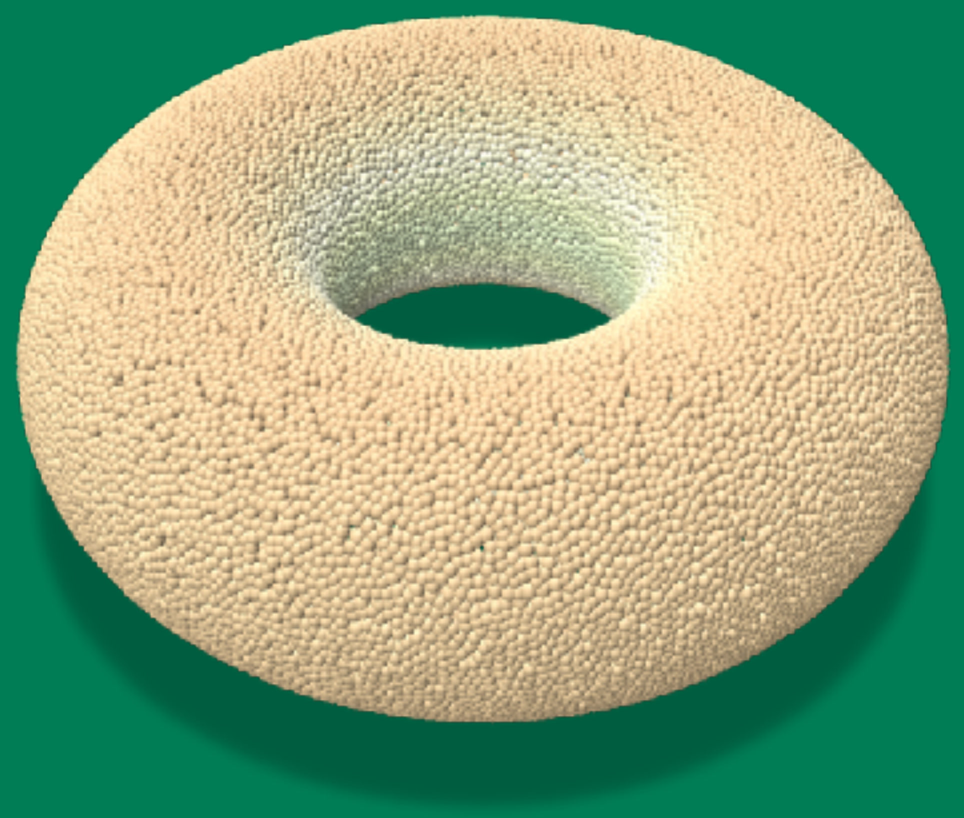
Exploring the Laplacian in Computer Graphics

Week 6

Crane He Chen

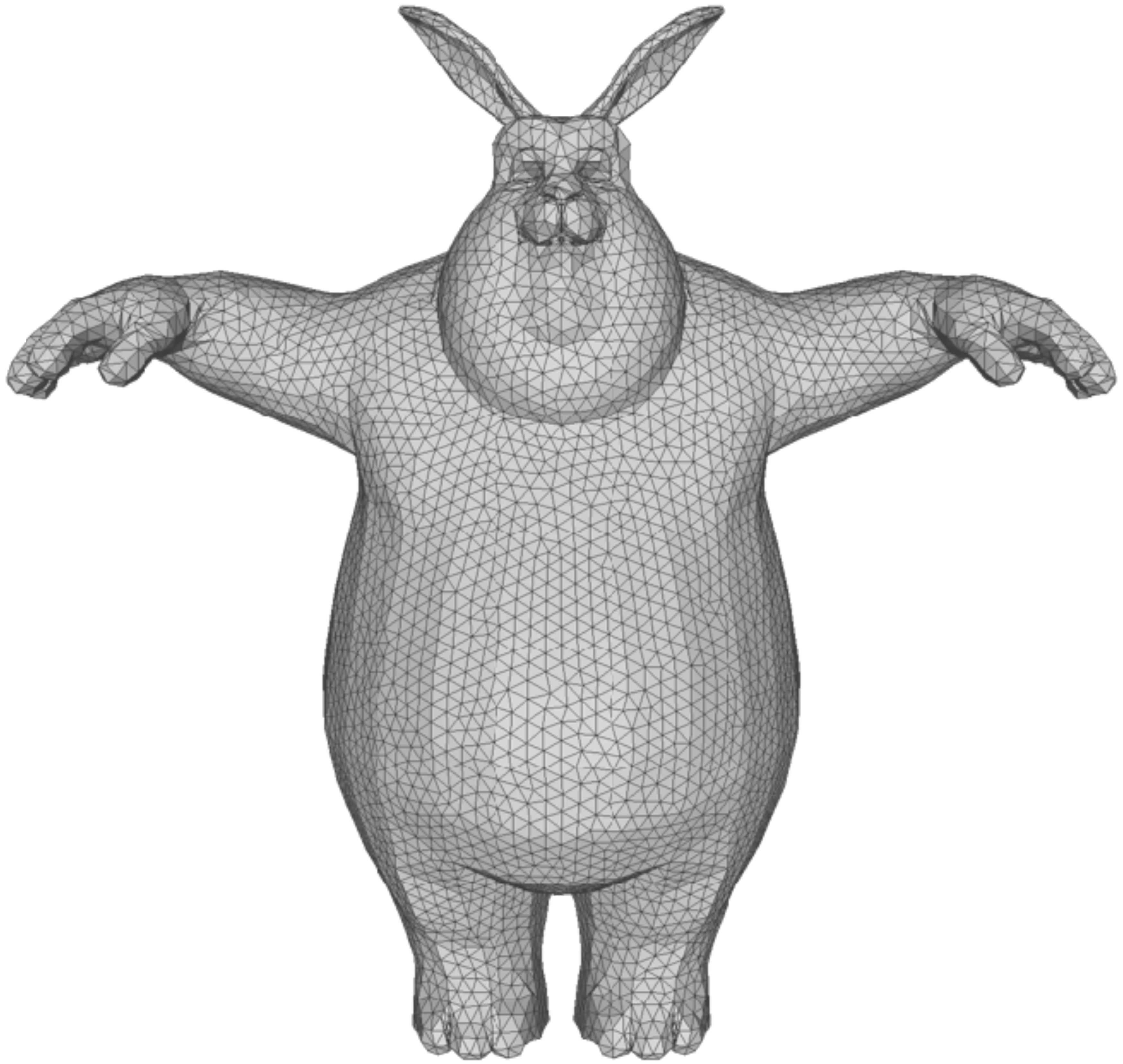
The Johns Hopkins University



2023 Fall

Review questions

Where could a function live on triangle meshes?

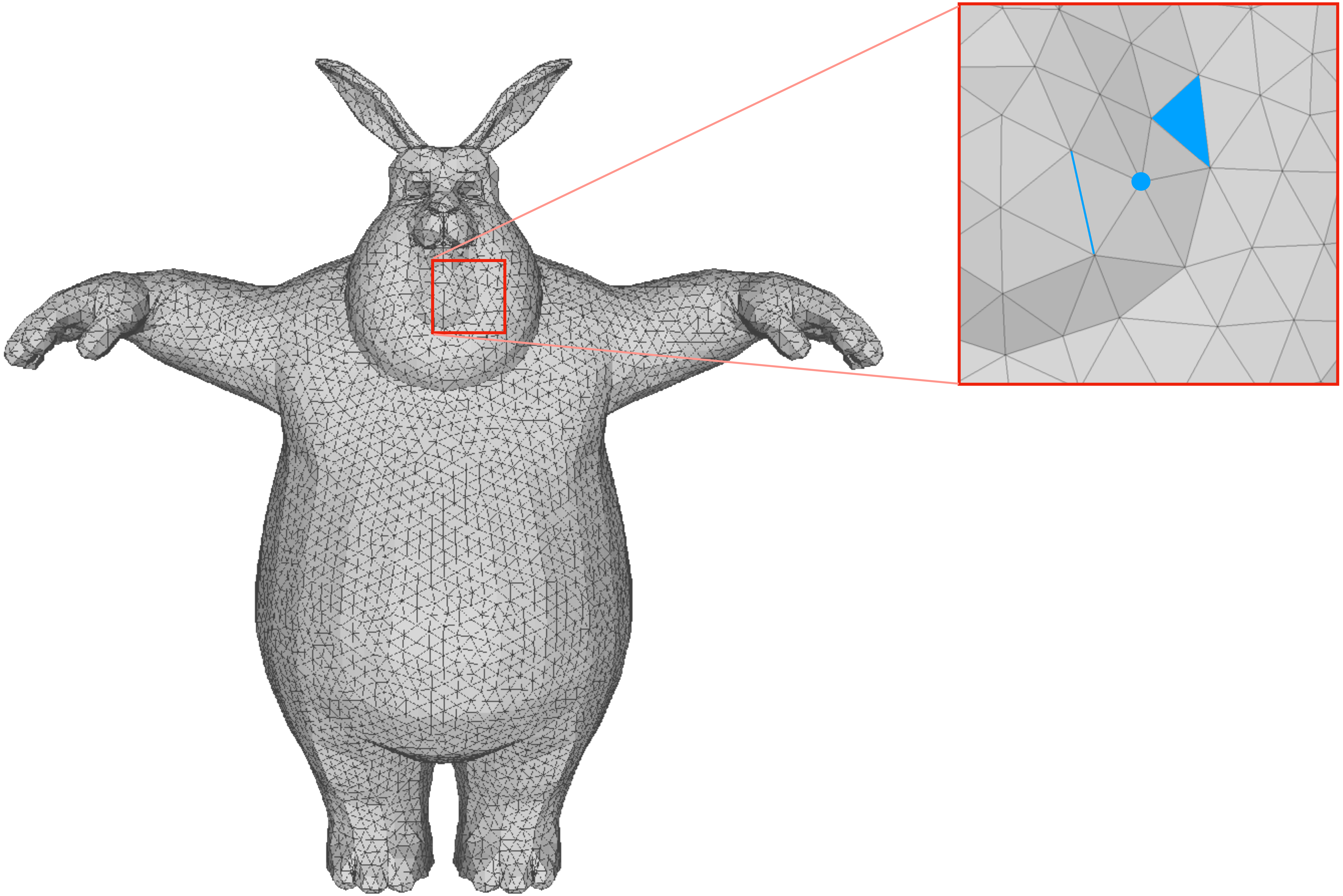


Answer

vertices

edges

faces



Review questions

What is the Laplacian?

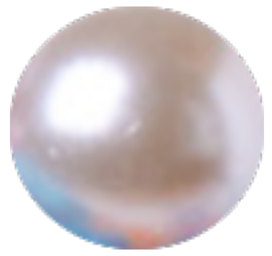
Answer

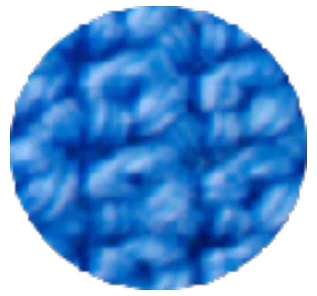
three dimensional (geometry processing)

scalar function living on vertices: beads

Laplacian: helps us understand how each bead is different from its neighboring beads

“quality inspector”
of beads distribution





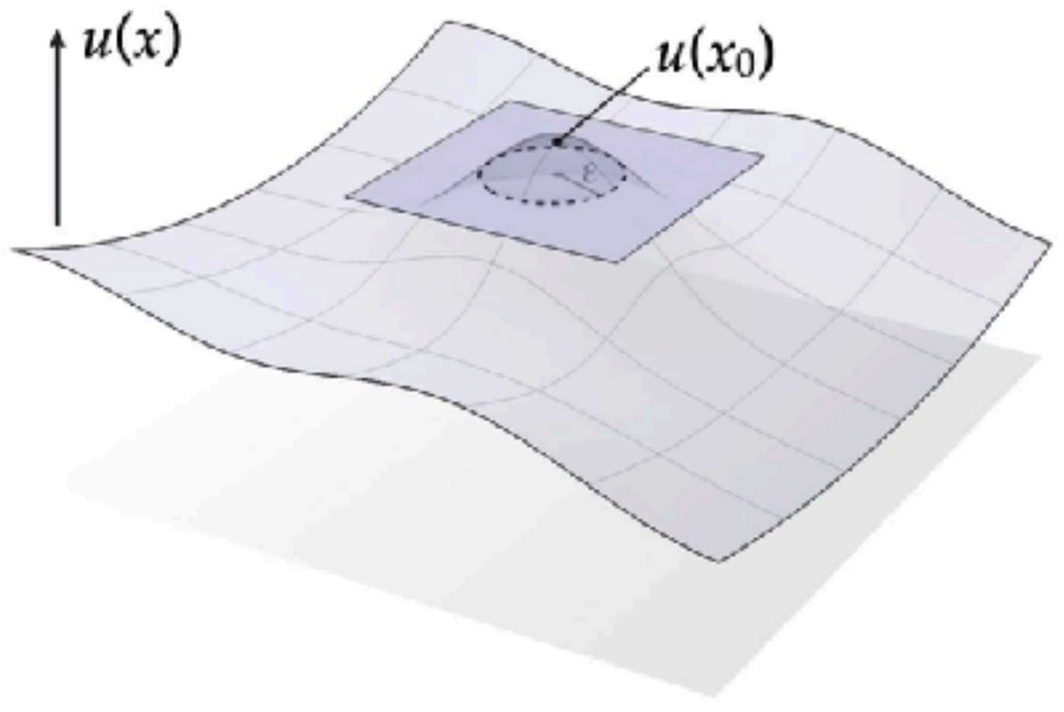
beads
(the scalar function)

“bridge”

knit
(the shape)

Answer

- Deviation from local average



- Sum of second derivatives

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

- Divergence of gradient

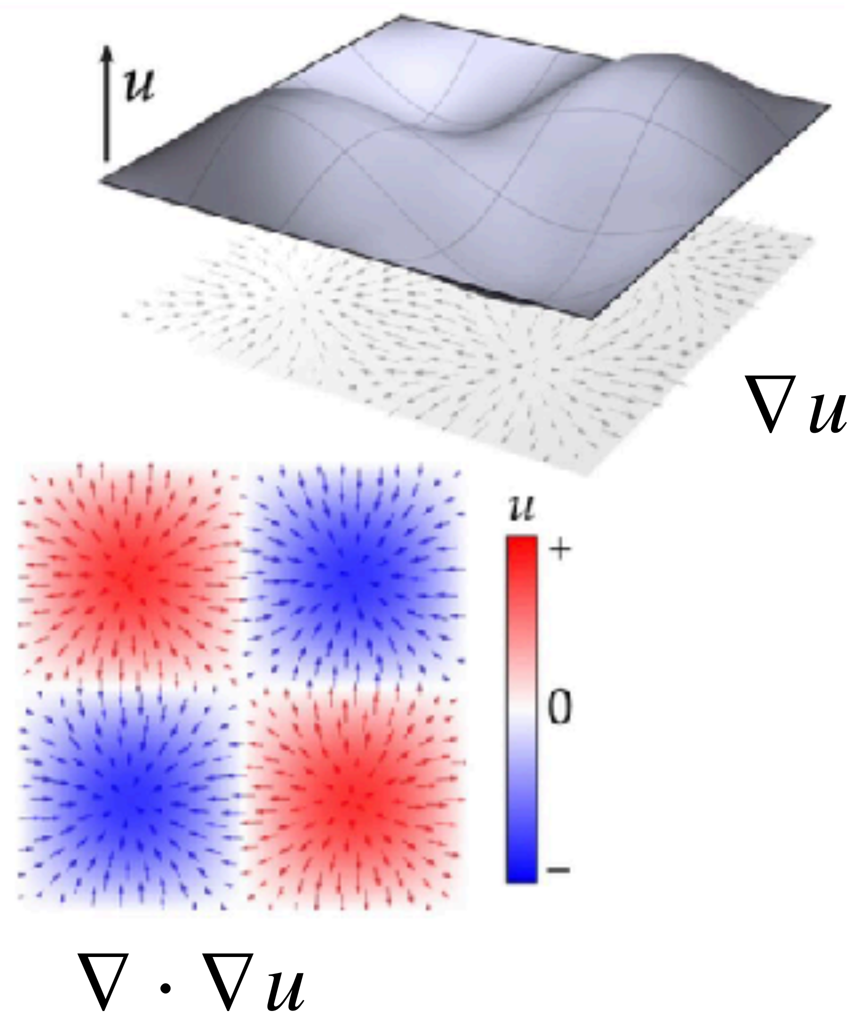


Figure stolen from Keenan Crane slides

Review questions

What is the other name of “the Laplacian”?
How are they slightly different from each other?

Answer

Δ **"The Laplacian"**

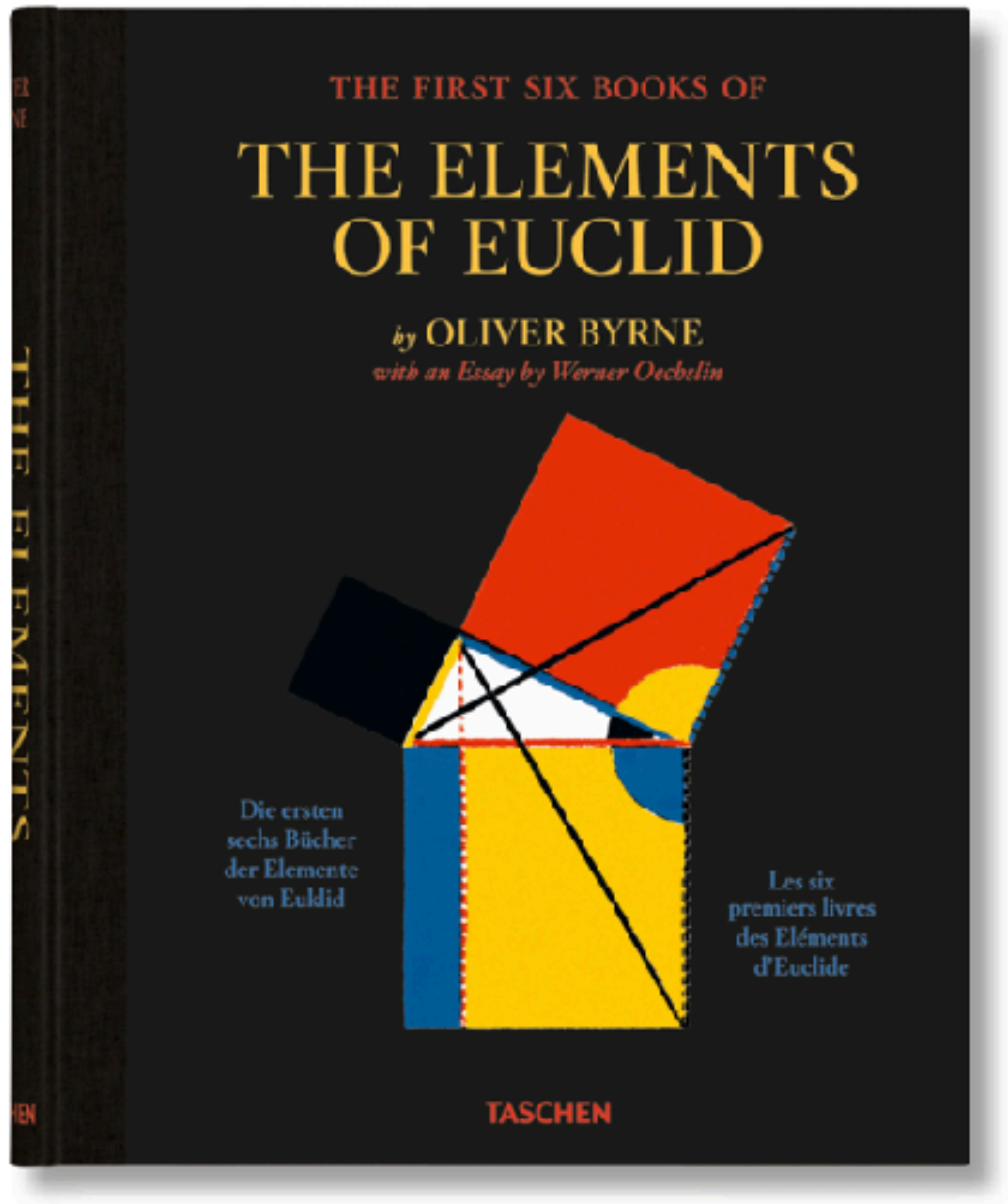
(Euclidean domain)

Δ **"Laplace-Beltrami Operator"**

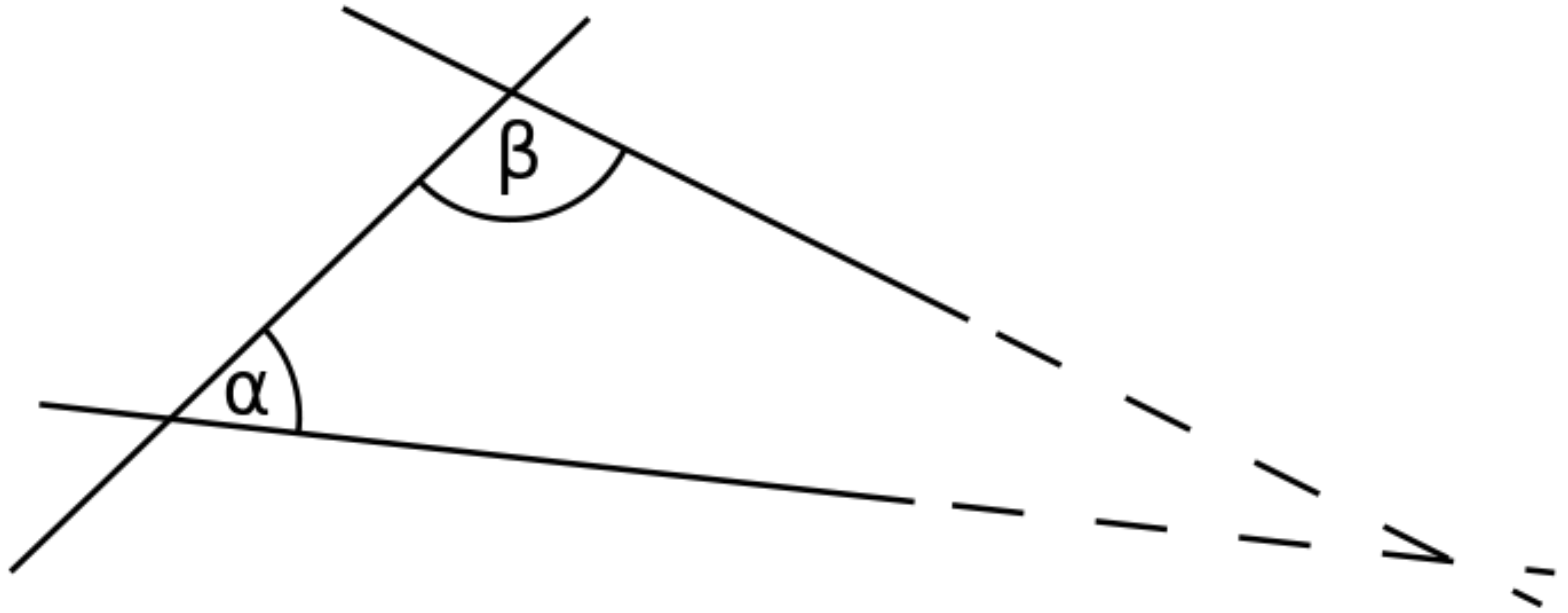
(curved domain)

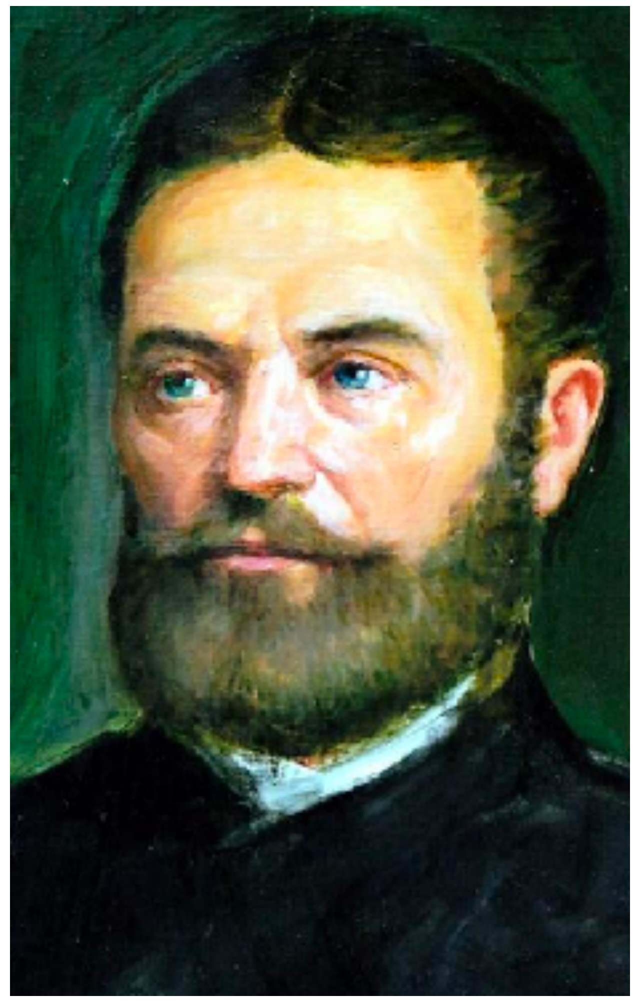
Answer

High school: Euclidean space

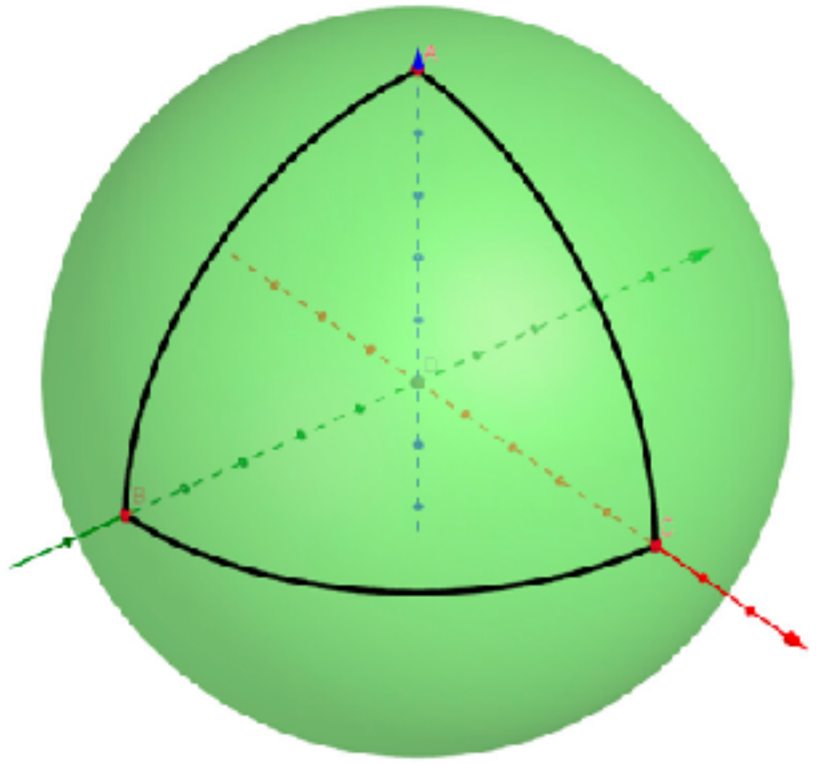


Euclid's postulate 5:





János Bolyai



When two straight lines intersect with a line segment, if the sum of the interior angles alpha and beta is less than 180, the two straight lines meet on that side.

What is Dirichlet energy?

Answer

Dirichlet energy $$\int_{\Omega} \|\nabla g(p)\|^2 dp$$

Minimizing Dirichlet energy (as smooth as possible)

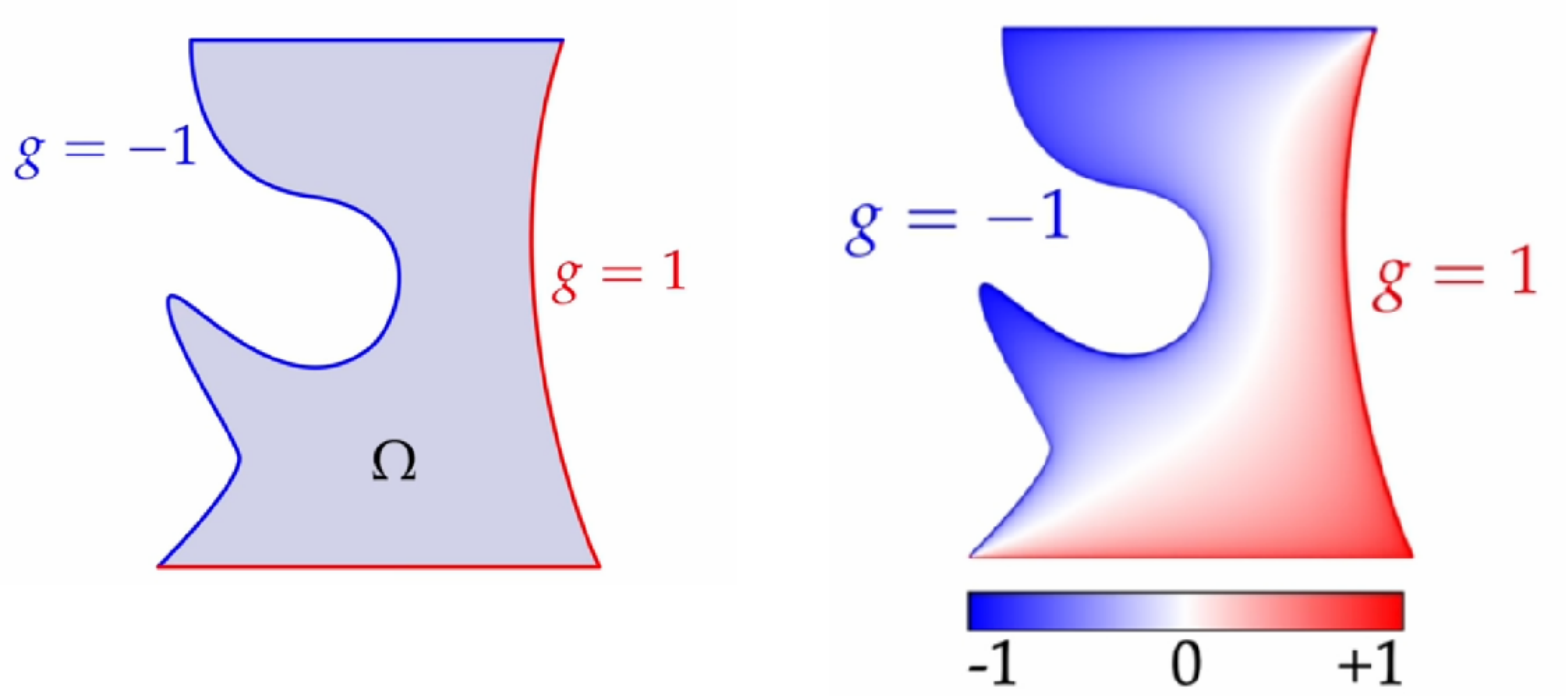
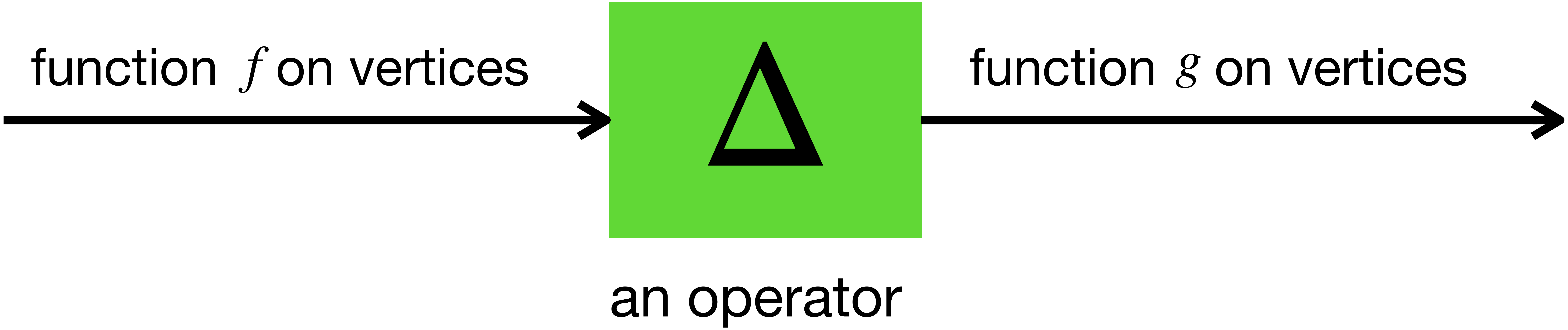


Figure stolen from Keenan Crane slides

Our Goal Today

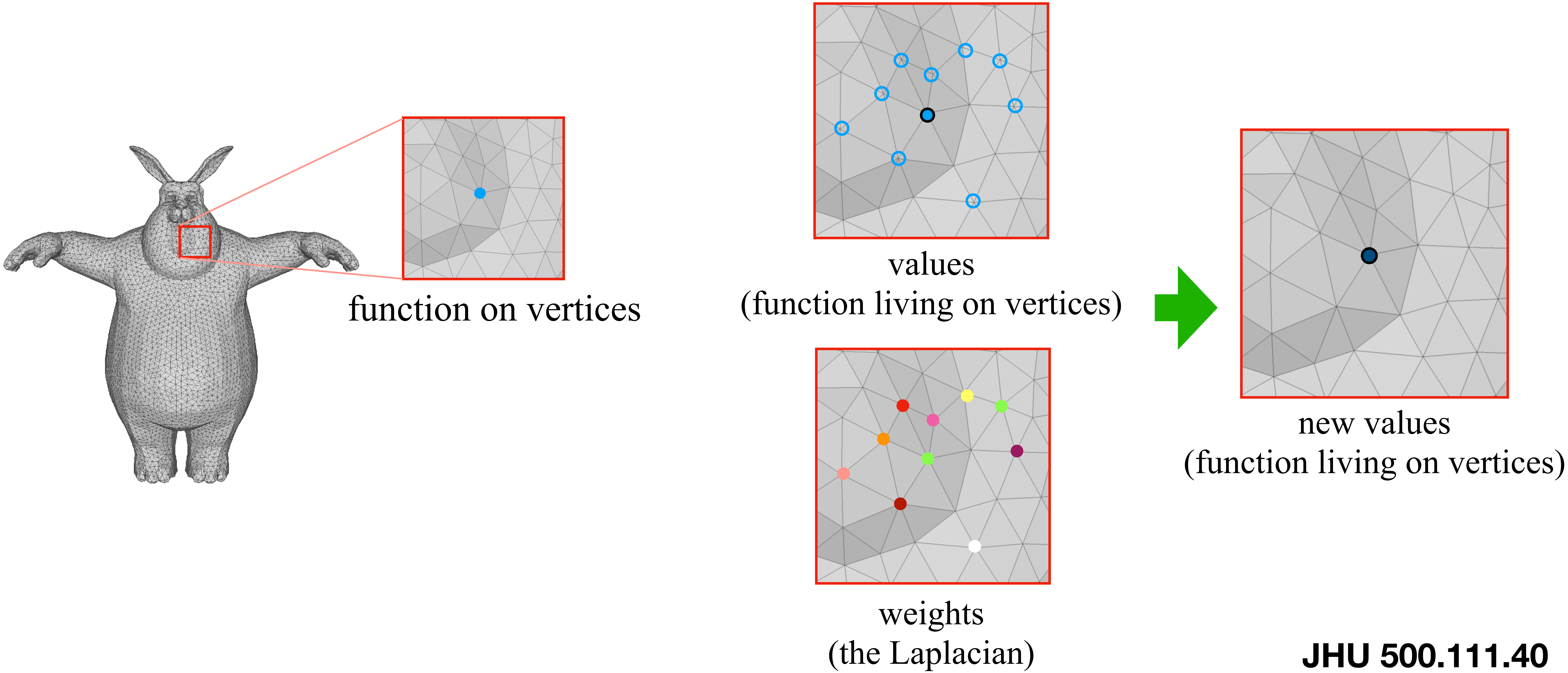
Calculate the Laplacian on Triangle Meshes

Discrete Laplacian



Discrete Laplacian

Options 1 to think about it (weights on vertices)



Discrete Laplacian

Options 1 to think about it (weights on vertices)

If there are N vertices on a triangle mesh,
the Laplacian can be represented by a NxN table ("matrix") L

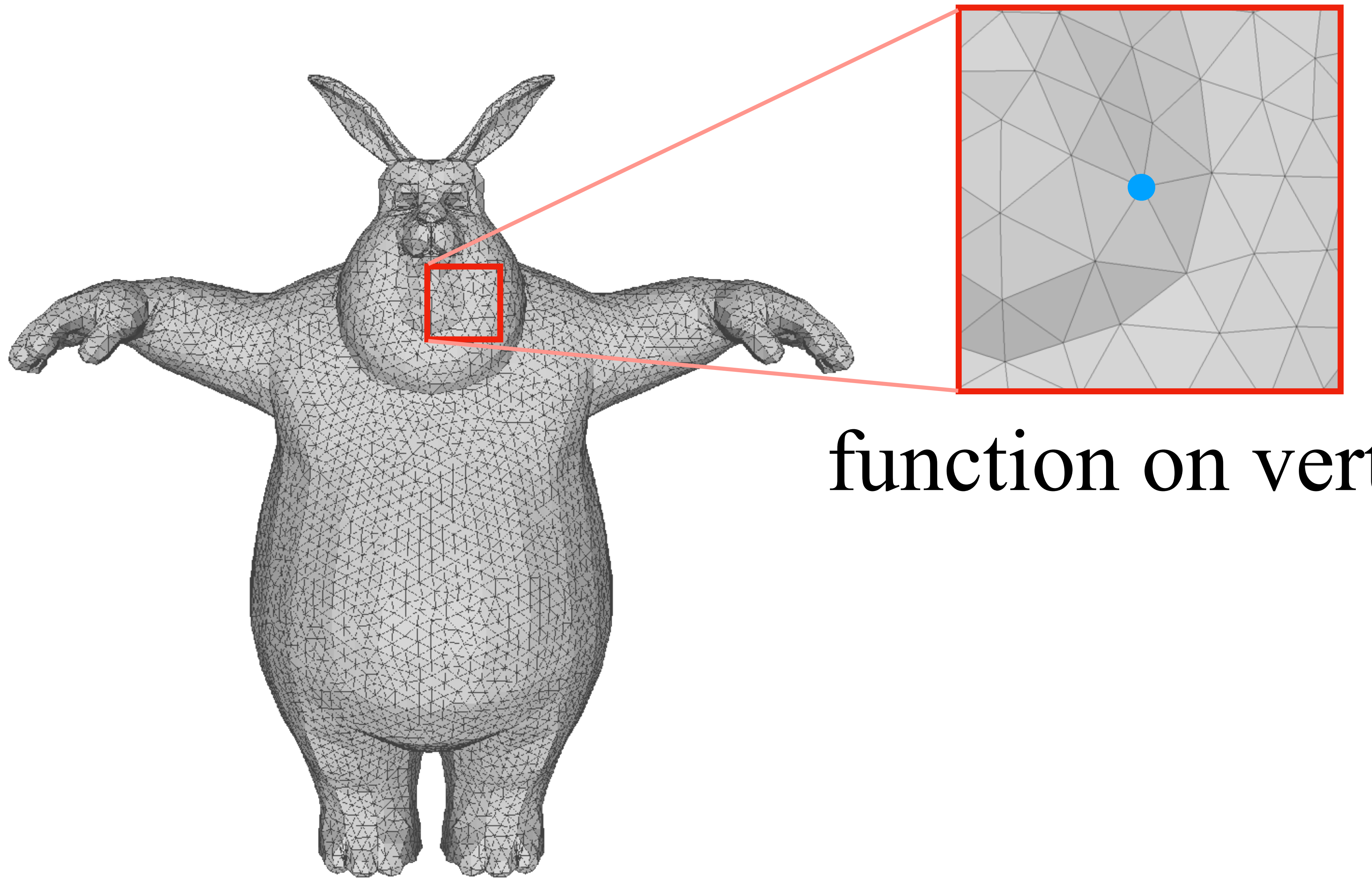
L_{ij} is row i, column j of this table

Applying the Laplacian on function f, we get another function g, where

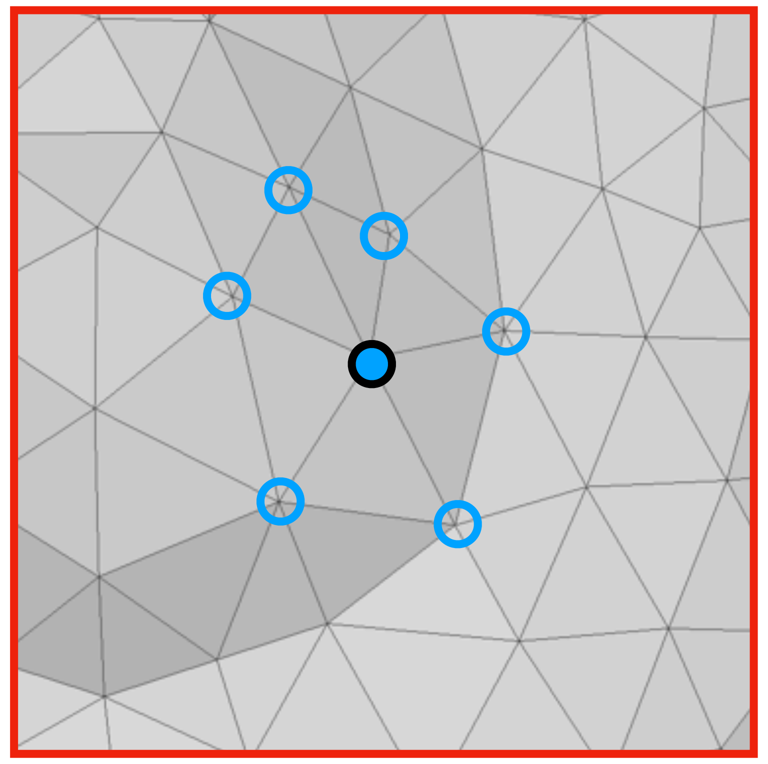
$$g_j = \sum_{v_j \in V} L_{ij} f_j$$

Discrete Laplacian

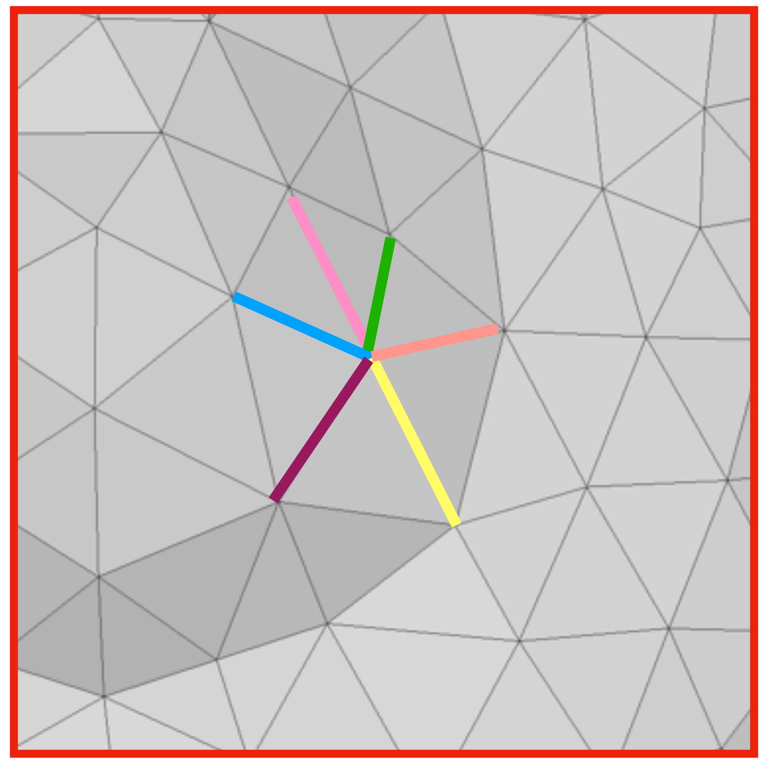
Options 2 to think about it (weights on edges)



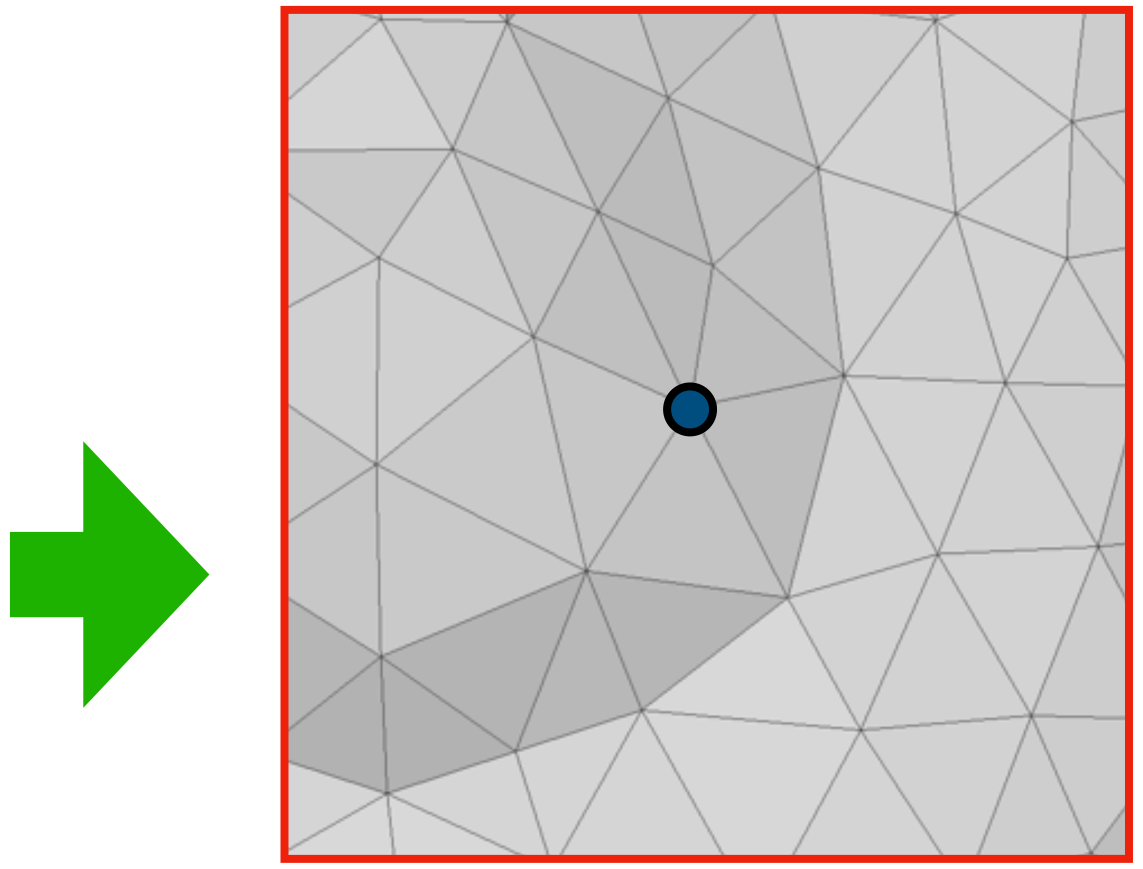
function on vertices



values
(function living on vertices)



weights
(the Laplacian)



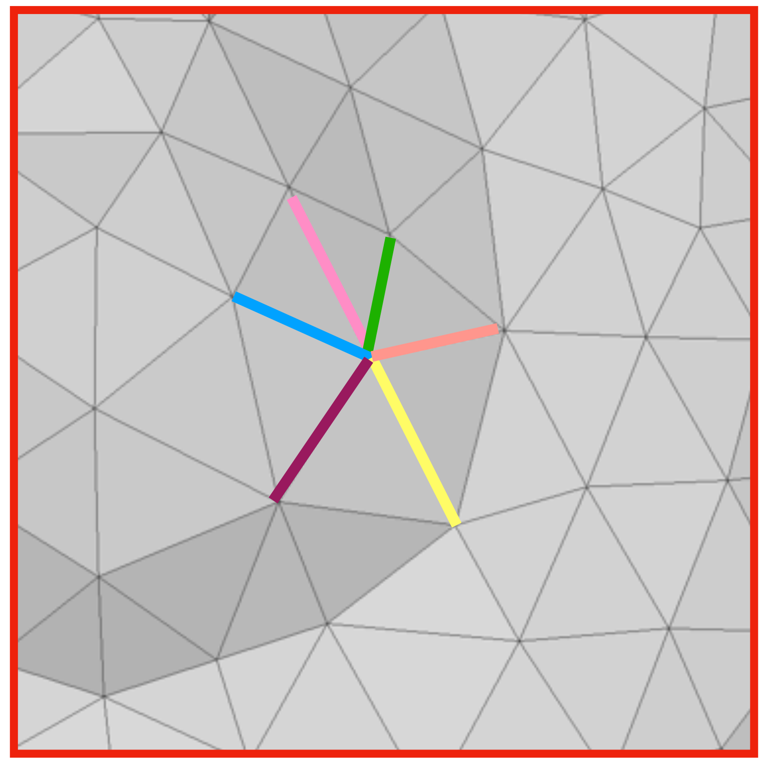
new values
(function living on vertices)

How to calculate the Laplacian?

Options 2 to think about it (weights on edges)

Applying the Laplacian on function f, we get another function g, where

$$g_j = \sum_{v_j \in Nbr(v_i)} L_{ij}(f_j - f_i)$$



weights
(the Laplacian)

Brainstorm:
What are possible ways to define these edge weights?

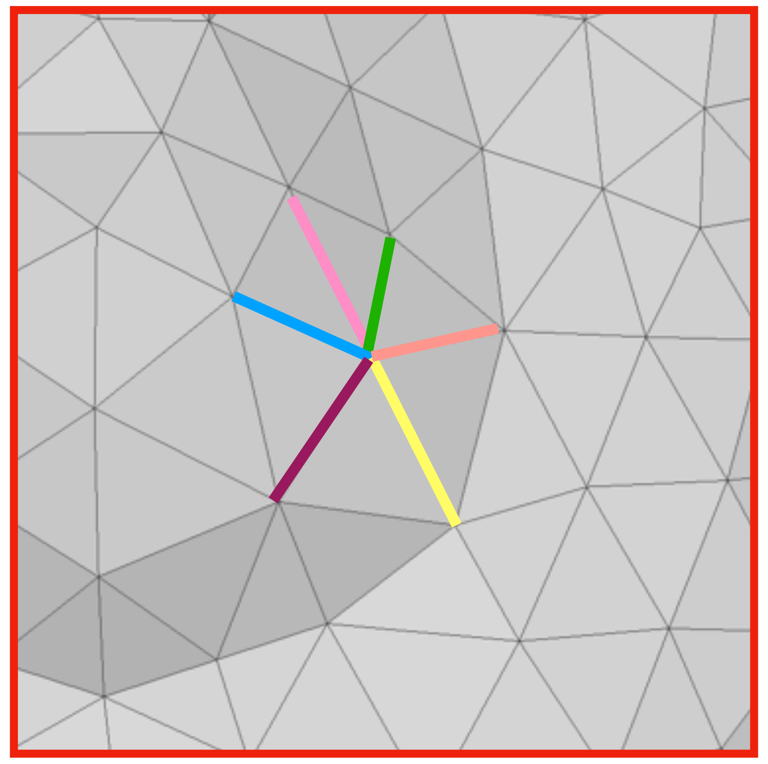
hint: graph theory

How to calculate the Laplacian?

Options 2 to think about it (weights on edges)

Applying the Laplacian on function f, we get another function g, where

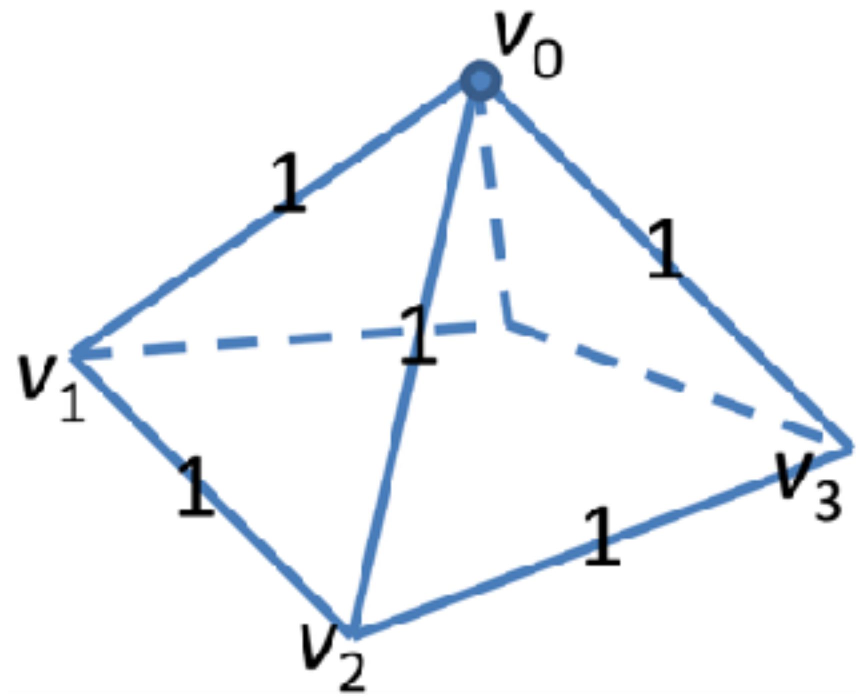
$$g_j = \sum_{v_j \in Nbr(v_i)} L_{ij}(f_j - f_i)$$



weights
(the Laplacian)

Tutte Laplacian:

$$L_{ij} = 1$$



Graph Laplacian:

$$L_{ij} = 1/\textbf{valence}$$

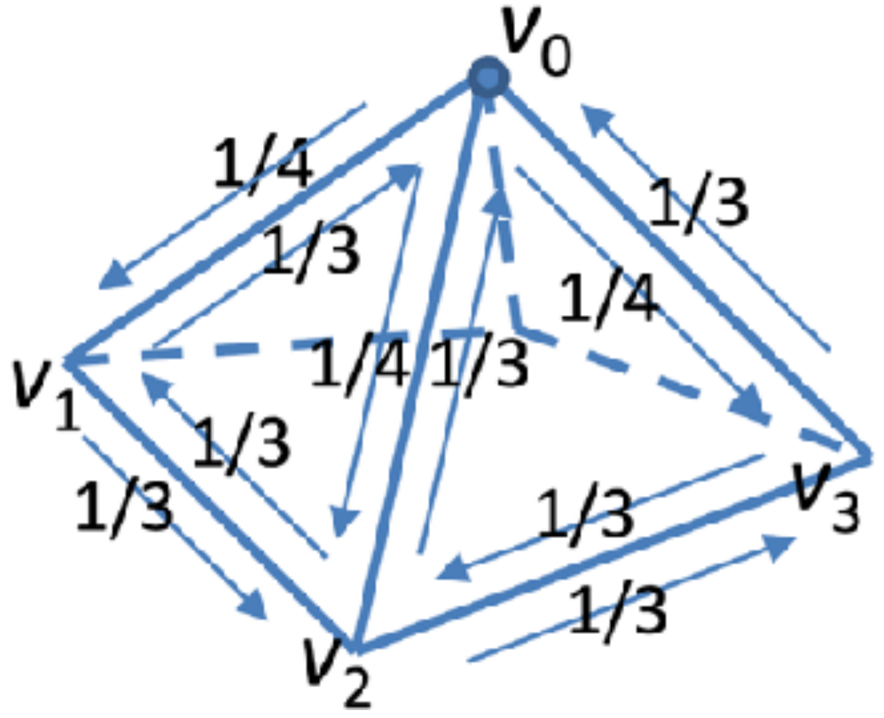


Figure stolen from Misha Kazhdan slides

How to calculate the Laplacian?

Options 2 to think about it (weights on edges)

Cotangent Laplacian (most widely used)

$$L_{ij} = \begin{cases} \frac{1}{2}\left(\cot\left(\alpha_{ij}\right) + \cot\left(\beta_{ij}\right)\right) & \text{if } i \neq j \text{ and } v_j \in \mathrm{Nbr}(v_i) \\ -\sum\limits_{v_k \in \mathrm{Nbr}(v_i)} L_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

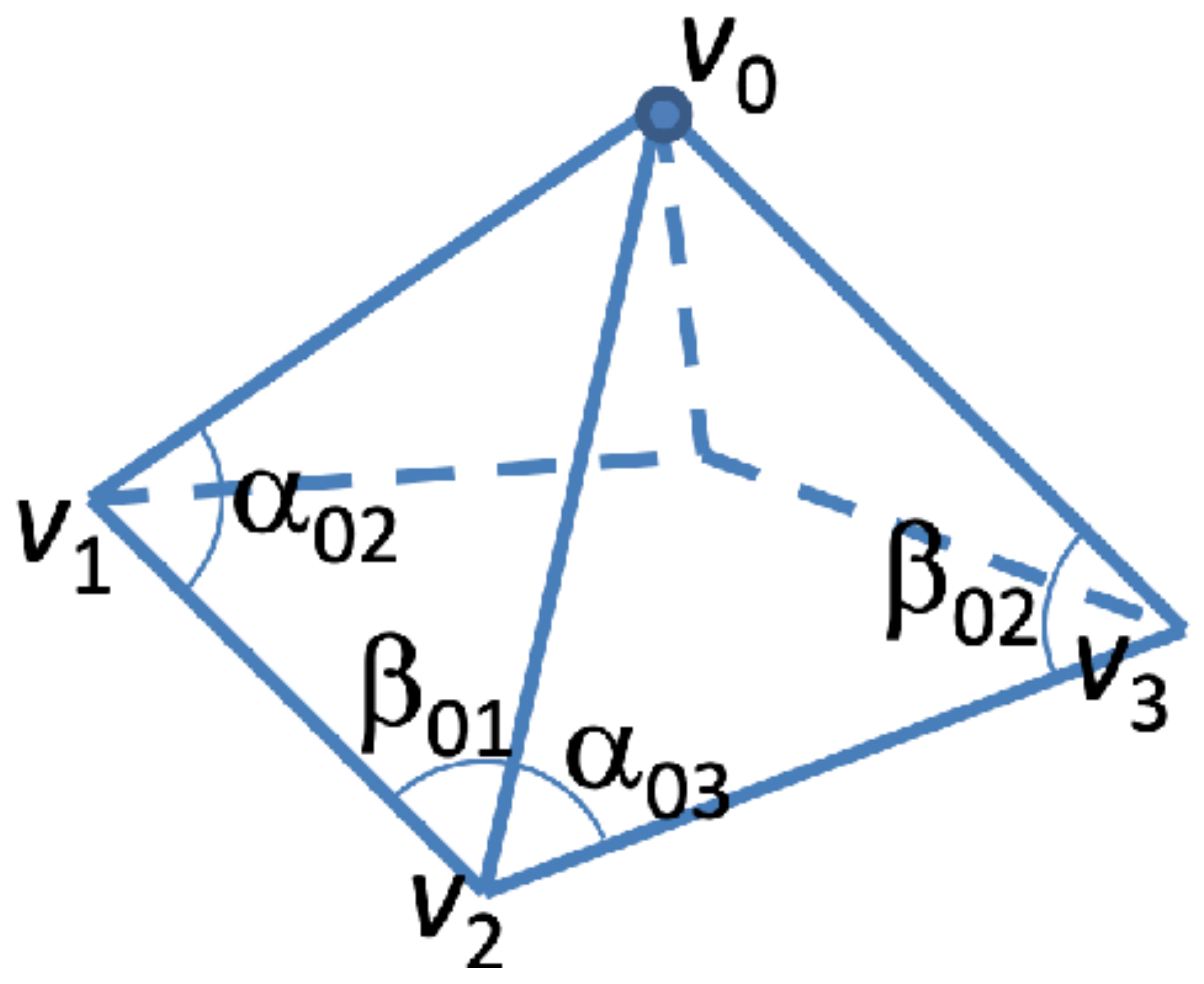


Figure stolen from Misha Kazhdan slides

Derive the cotangent Laplacian

Continuous case

pinching the rubber sheet

1. Pinching introduces a deformation
2. There is a local difference of the height map (the Laplacian of height function)
3. Some local grid squares are stretched, others are compressed, in other words, the change of area (gradient of area)
4. They are both describing the deformation and has a connection

https://cacm.acm.org/news/163925-a-touchscreen-you-can-pinch-poke-and-stretch/fulltext?mobile=false

Derive the cotangent Laplacian

Continuous case

pinching the rubber sheet

If the value of the function at vertex v is the position of v, then the Laplacian of the function at v should be the area gradient.

https://cacm.acm.org/news/163925-a-touchscreen-you-can-pinch-poke-and-stretch/fulltext?mobile=false

Derive the cotangent Laplacian

Discrete case

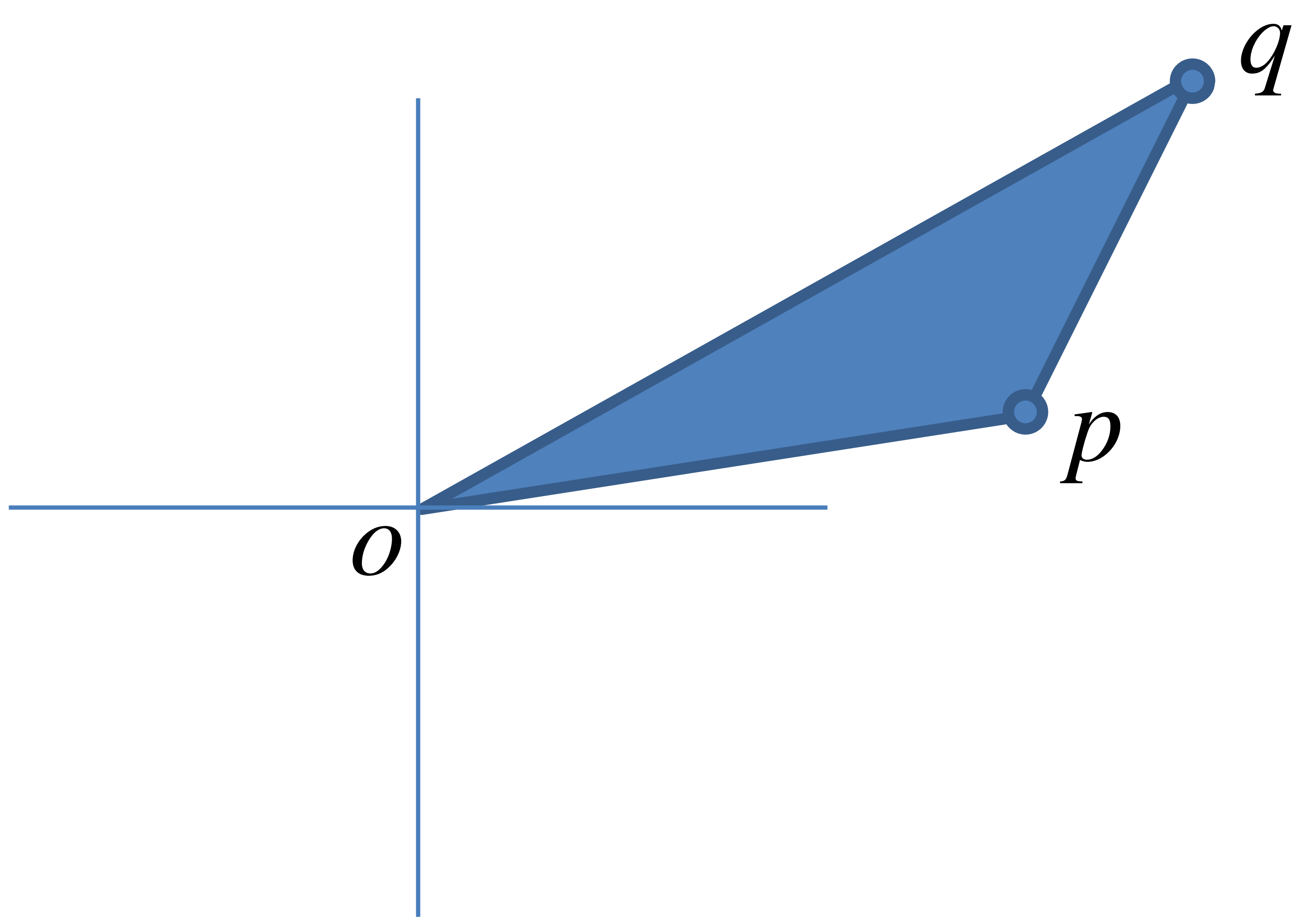
Desirable properties for the discrete Laplacian

- Sparsity (only neighbors, no effect from distant vertices)
- Positivity (when averaging neighboring values, we want non-negative weights)
- Symmetry (this inherits from continuous setting)
- Linear Precision (If the mesh lives in a plane and the function values are obtained by sampling a linear function, the Laplacian of the function should be zero.)

Derive the cotangent Laplacian

Discrete case

Given a triangle (o, p, q), what direction should we move o, in order to maximally increase the area?



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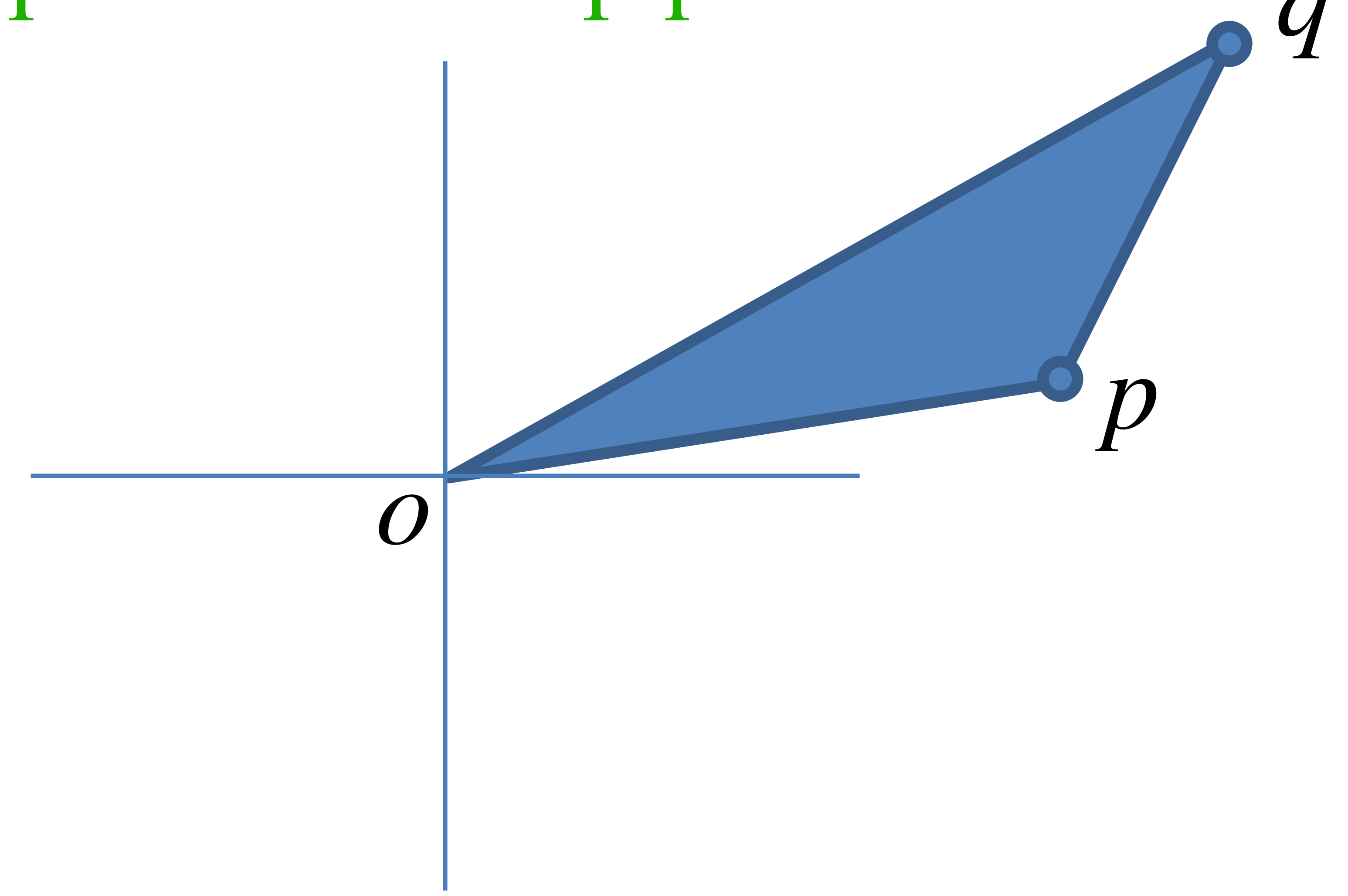
Derive the cotangent Laplacian

Discrete case

Given a triangle (o, p, q), what direction should we move o, in order to maximally increase the area?

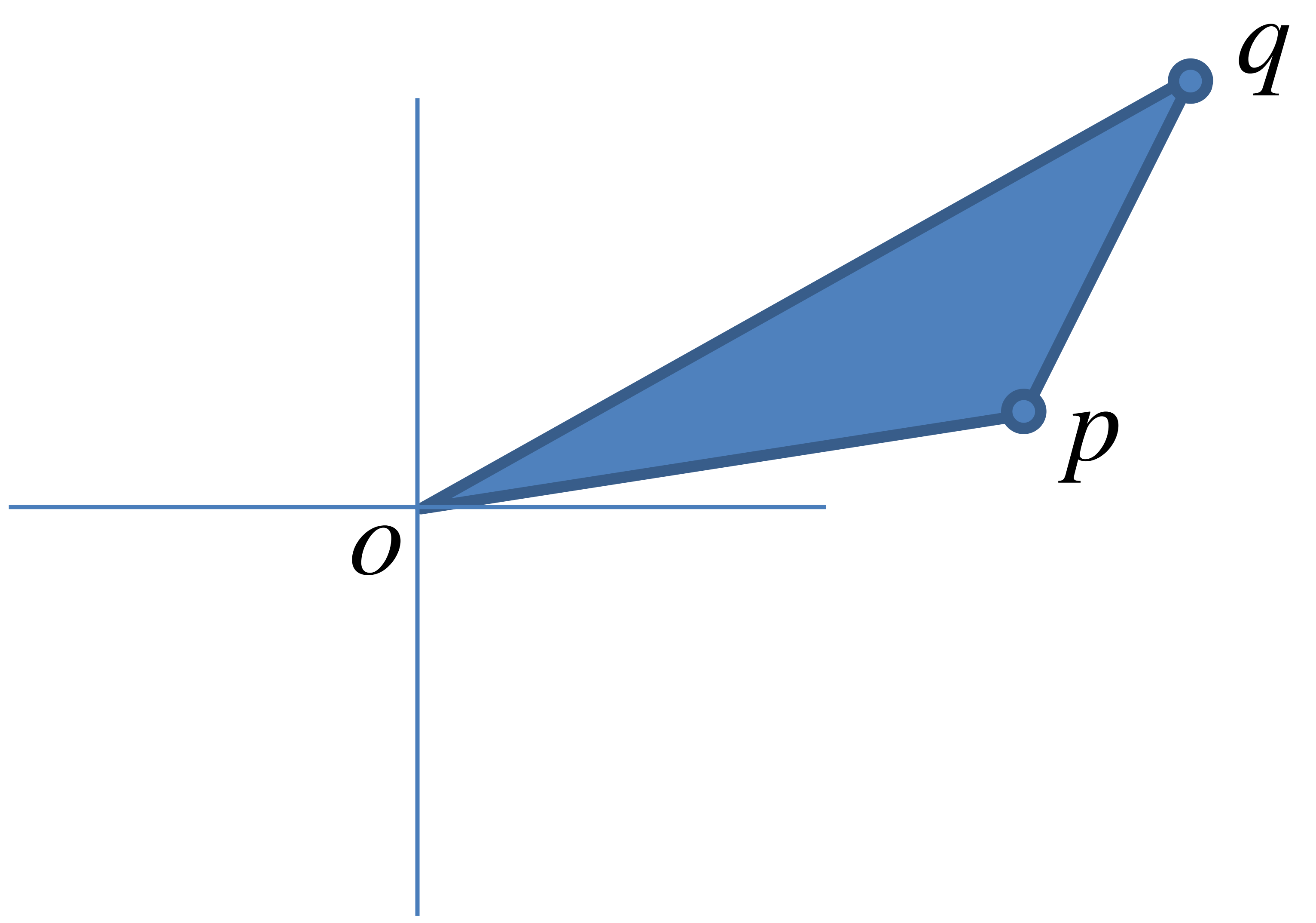
The area of triangle is half base times height.
If we set pq as the base, we would want o to move in the direction perpendicular to pq.



Derive the cotangent Laplacian

Discrete case

If we take a step size eps in that direction, how will the area change?



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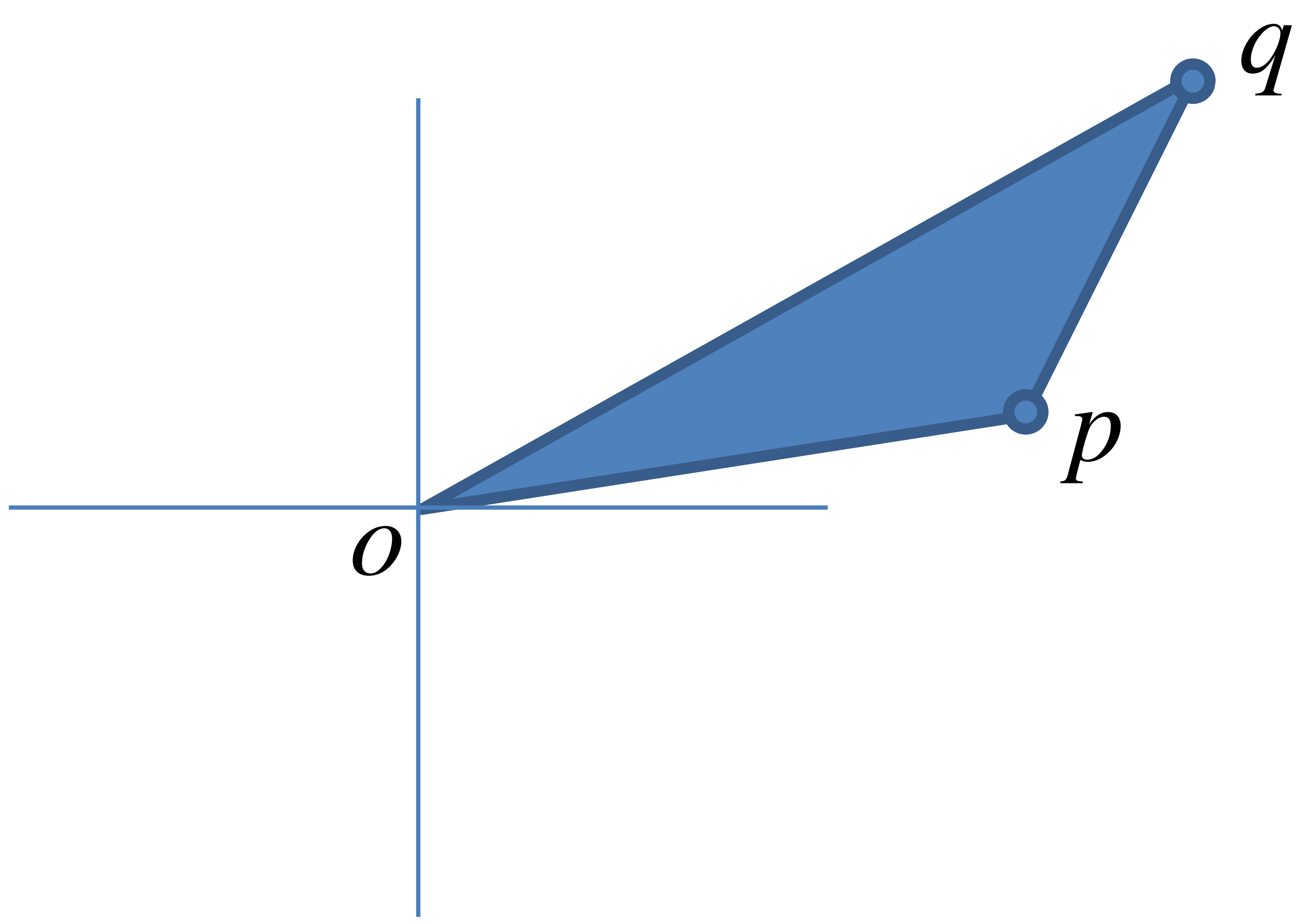
Derive the cotangent Laplacian

Discrete case

If we take a step size ϵ in that direction, how will the area change?

$\epsilon|p-q|/2$



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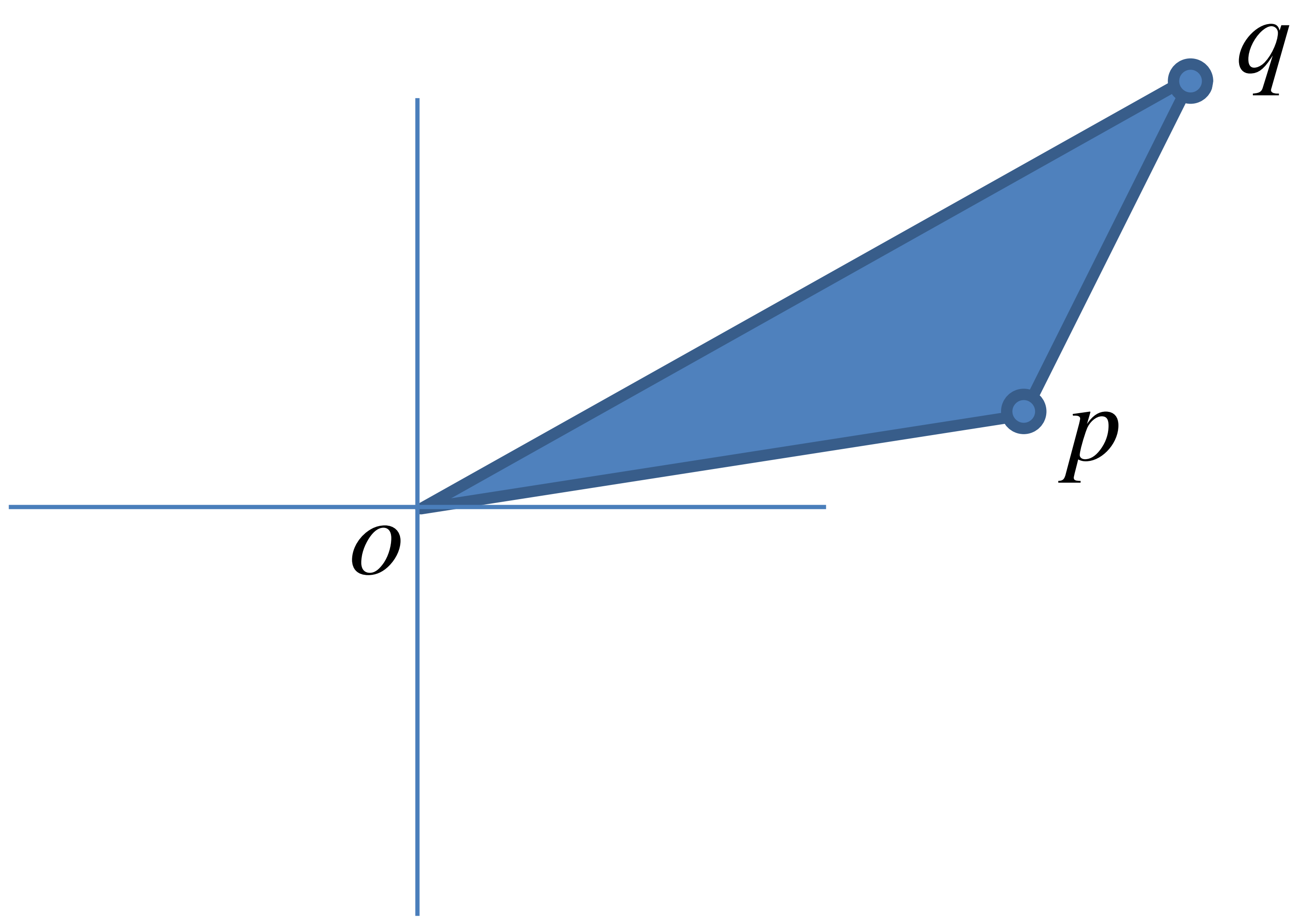
Derive the cotangent Laplacian

Discrete case

Therefore, the gradient is the vector perpendicular to pq, with the length |p-q|/2.



Derive the cotangent Laplacian

Discrete case

The area of triangle is half base times height.
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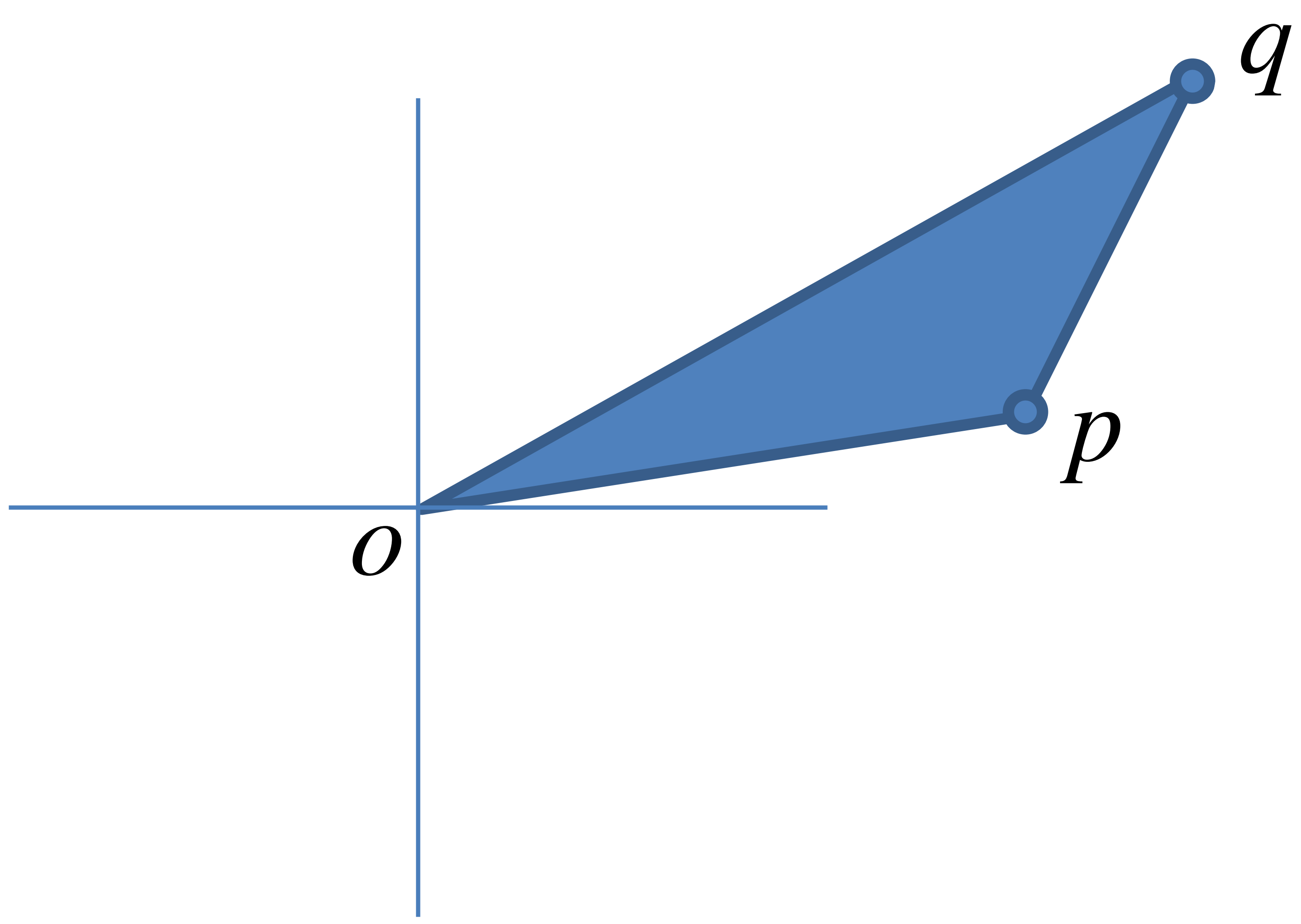

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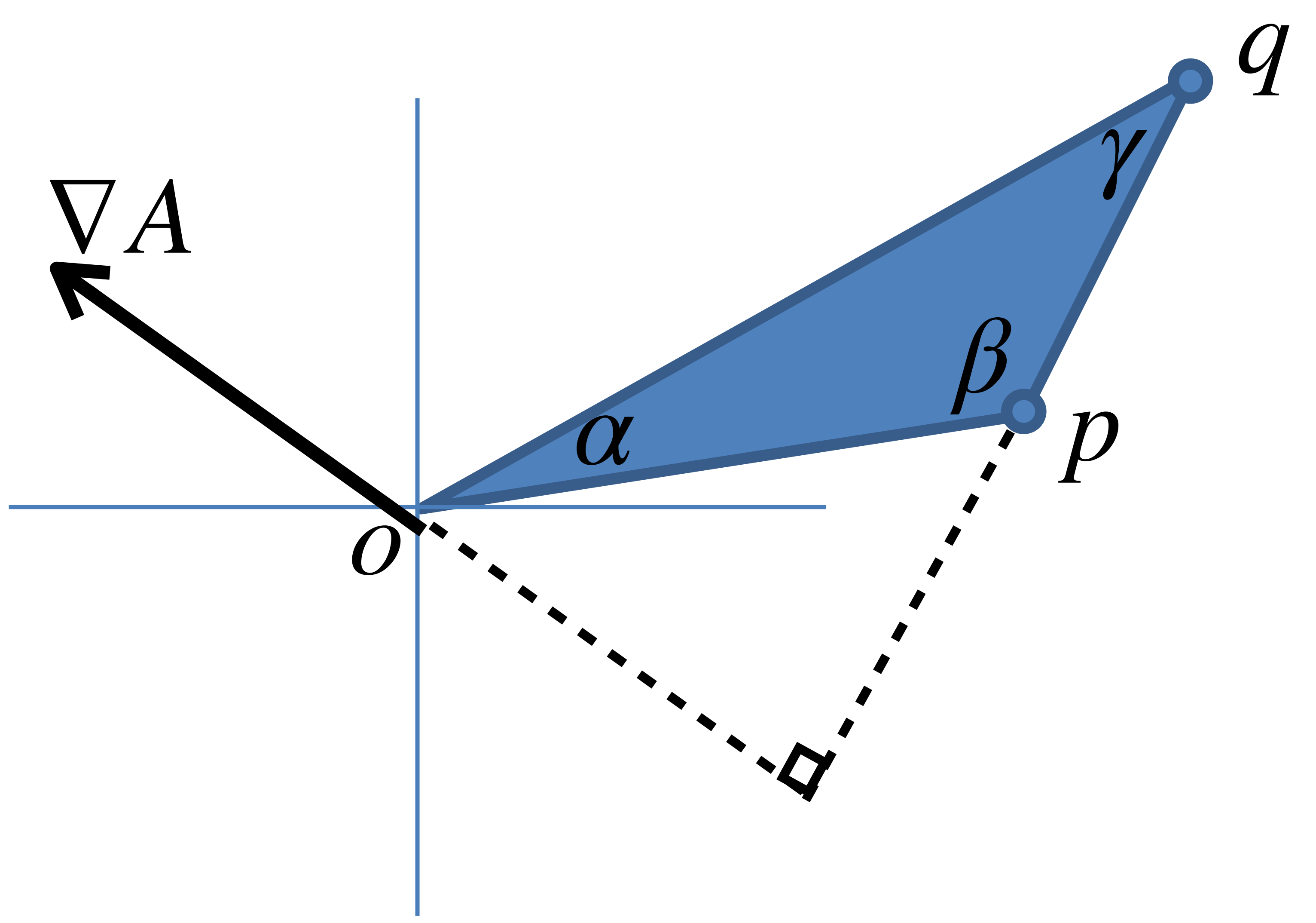
Derive the cotangent Laplacian

Discrete case

What's the vector perpendicular to p-q and with length |p-q|/2?

After derivations (details given at the "Other Resources"), we have

$$\frac{1}{2}(cot(\beta)p + cot(\gamma)q)$$



Derive the cotangent Laplacian

Discrete case

This leads to the cotangent Laplacian:

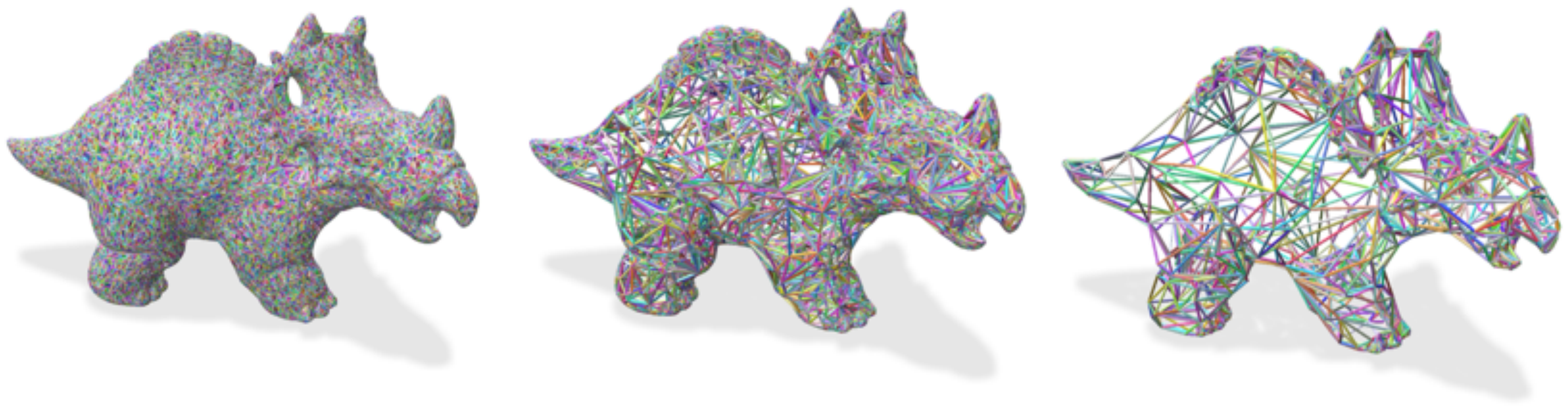
$$L_{ij} = \begin{cases} \frac{1}{2}(cot(\alpha_{ij}) + cot(\beta)_{ij}) & \text{if } i \neq j, v_j \in Nbr(v_i) \\ -\sum L_{ik} & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

Take-aways from Today's Lecture

- You learned two options to understand Laplacian as "weights"
- You learned Tutte Laplacian, Graph Laplacian
- You learned cotangent Laplacian

More gears for art contest!

decimation



swept volume

cubic stylization

https://www.dgp.toronto.edu/projects/swept-volumes/

https://www.dgp.toronto.edu/projects/cubic-stylization/

https://github.com/HeCraneChen/curvature-qslim-mesh-decimation

Now, your turn!

Go to the course Github page to download code!

We’ll wok on coloring the bunny together!

Pair-Coding

```
int main(int argc, char *argv[])
{
  using namespace Eigen;
  using namespace std;

  // variable definition
  Eigen::MatrixXd V, PD1, PD2, PV1, PV2;
  Eigen::MatrixXi F;
  Eigen::VectorXd total_curvature, total_curvature_vis;

  // calculate total curvature
  igl::read_triangle_mesh("../data/BigBuckBunny.ply",V,F);
  igl::principal_curvature(V, F, PD1, PD2, PV1, PV2);
  total_curvature = PV1.array().square() + PV2.array().square();
  total_curvature_vis = total_curvature.array().pow(0.01);

  // visualization
  polyscope::init();
  polyscope::options::groundPlaneMode = polyscope::GroundPlaneMode::ShadowOnly;
  auto psMesh = polyscope::registerSurfaceMesh("bunny", V, F);
  auto TotalCurvature = polyscope::getSurfaceMesh("bunny");
  auto ScalarQuantity1 = TotalCurvature->addVertexScalarQuantity("TotalCurvature", total_curvature_vis);
  ScalarQuantity1->setColorMap("jet");
  ScalarQuantity1->setEnabled(true);
  polyscope::options::shadowDarkness = 0.1;
  polyscope::show();

}
```

https://github.com/HeCraneChen/500.111.40_exploring_laplacian_graphics_course/tree/main/week3

Are There Any Questions?